Strong and weak turbulence face to face
Stably-stratified, rotating, 3D flows

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Outline

- A General context in 3D turbulence in fluids with waves. Non-propagating modes coexisting with dispersive wave modes.

- Incorporating wave-turbulence theory in a “triadic” closure for strong turbulence (e.g. fully anisotropic multimodal EDQNM developed on linear eigenmodes for fluctuations.)

- The toroidal cascade face to gravity waves turbulence in stably-stratified turbulence

- New recent results for decoupling the 2D non-propagating mode in axially confined rotating turbulence (Julian F. Scott, J. Fluid Mech., to appear)
Identifying non-propagating modes and wave-modes

- The basic eigenmodes decomposition prior to Wave turbulence theory,
  \[ \hat{\nu} = a_0(k, t)N^{(0)} + a_1(k, t)N^{(1)}e^{i\sigma_k t} + a_{-1}(k, t)N^{(-1)}e^{-i\sigma_k t}, \]
  to be reinjected in basic Navier-Stokes-Boussinesq-type eqs, with \( \sigma_k \) the dispersion law (continuous 3D wave-space)

- Exemple of the 0-mode in 3D unbounded (statistically homogeneous) turbulence:
  toroidal (+ VSHF) mode in stably-stratified turbulence, QG mode in rotating stably-stratified turbulence, solenoidal mode in weakly compressible turbulence at low Mach number (e.g. Sagaut & Cambon book, 2008 and references therein)

- A low dimension zero-mode, at \( \sigma_k = 0 \), or a full zero-mode? Unbounded versus confined rotating turbulence (Julian Scott, JFM, to appear). Results using conventional pseudo-spectral DNS are ambiguous.
Exact equations for “slow” amplitudes

- Linear limit: dispersion law $\sigma_k$, eigenmodes $N^s(\pm k)$, $s = 0, \pm 1$, $a_0, a_{\pm}$ are just constant. Nonlinear?

- From basic Navier-Stokes-type equations

$$\dot{a}_s(k, t) = \sum_{s', s'' = 0, \pm 1} \int G_{kpq}^{ss'}e^{i(s\sigma_k + s'\sigma_p + s''\sigma_q)t}a_{s'}(p, t)a_{s''}(q, t)d^3p,$$

with $s = 0, \pm 1$, $k + p + q = 0$.

Classical approach of Wave Turbulence, two time-scales $t$ and $\epsilon t$, possibly incorporated in EDQNM (e.g. “EDQNM3” transferring the machinery of EDQNM from $\hat{u}$ to slow amplitudes.)
Rotating stratified flows: governing equations

\[
\frac{\partial u_i}{\partial t} + f \epsilon_{i3j} u_j - b \delta_{i3} + \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0
\]

\[
\frac{\partial b}{\partial t} + N^2 u_3 = P_r \nu \nabla^2 b - u_j \frac{\partial b}{\partial x_j}
\]

2 external constant parameters \( N \) (vertical stabilizing stratification frequency) and \( f \) (Coriolis parameter), \( \sigma_k^2 = N^2 \sin^2 \theta_k + f^2 \cos^2 \theta_k \)

Valid for a liquid or a gas. \( P_r \) characterizes the diffusivity of the stratifying agent (temperature, salt)
512^3 DNS, Liechtenstein et al. 2005, without mean shear

\[ 2\Omega = f = N \]
Wave aspects in rotating and/or stratified turbulence

Left, stable stratification, right, rotation

Conical region in which energy concentrates
The toroidal cascade and beyond, $f = 0$
Angle-dependent toroidal and poloidal modes (Liechtenstein, 2006)
Figure 1: Craya-Herring frame \( (e^{(1)}, e^{(2)}, e^{(3)}) \) in Fourier space.
Disentangling purely toroidal cascade from wave related cascade, as
\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \mathcal{E}^{(\text{tor})}(k, \cos \theta_k, t) =
\]
\[
\langle u^{(1)}(k, t) e^{(1)}(k) \rangle \cdot \left( \iiint_{p+q=k} \nu p u^{(1)}(p, t) e^{(2)}(p) \times u^{(1)}(q, t) e^{(1)}(q) d^3 p \right) + \ldots
\]
(\text{other terms in which at least another component than } u^{(1)} = u^{(\text{tor})} \text{ is present}), so that
\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \mathcal{E}^{(\text{tor})}(k, \cos \theta_k, t) = T^{(\text{tor})}(k, \cos \theta_k, t)_{000} + Fr^\alpha T^{(\text{tor})}_{WT}
\]
(generalized Lin eqs. much more usefull and general than Kármán-Howarth- something eqs. !). Complete system of spectral Lin’s equations for toroidal energy, total wave energy, poloidal buoyancy flux, and pseudo-polarization (imbalance).
The toroidal cascade and beyond

- The toroidal mode partly decouples from gravity waves. This questions a priori global scalings in terms of Froude number(s): Hanazaki & Hunt (RDT), Lindborg, Chomaz, Billand, Brethouwer, Galtier and coworkers: Vertical Froude number never smaller than 1 (dogma ?!) . Coming back to “wave-vortex” Riley et al. 1981 is better, with possibly small vertical Froude number

- The toroidal cascade is a ‘strong’ cascade, vs. a ‘weak’ gravity-wave turbulence cascade

- It explains the layering (lasagna) even from an initially unstructured state, without need for artificial 2D horizontal forcing. Zig-zag instability suggests only a very particular modality of the layering.
From unbounded to axially bounded rotating turbulence

- The unbounded case, $N^s(k) = e^{(2)} - s\nu e^{(1)}$, $s = \pm 1$,
  \[ \sigma_k = 2\Omega \cos k, \Omega = 2\Omega \frac{k_{\parallel}}{k}, \]
  CC & Jacquin (JFM, 1989) to Bellet et al. (JFM, 2006). Saturated two-dimensionalization, no inverse cascade. The 2D mode $k_{\parallel} = 0$, zero wave-mode, is an integrable singularity embedded in a fully 3D problem.

- Axial confinement, slab of thickness $2L$, $k_{\perp}$ continuous, $k_{\parallel} = n\frac{2\pi}{L}$, Rigorous wave-turbulence (or AQNM) axisymmetric kinetic equations for a covariance matrix $A_{nm}(k_{\perp})$.
  - $n = 0, k_{\parallel} = 0$, is now the fully relevant 2D non-propagating mode, which decouples from the $n \neq 0$ modes. ($O(Ro^{-1})$)
  - $A_{nn}, n \neq 0$ is governed by classical WT theory ($O(Ro^{-2})$)
  - $A_{nm}, n \neq m \neq 0$, are damped/scattered by the 2D mode ($O(Ro^{-2})$).
• CC and Jacquin, *J. Fluid Mech.* 202, 1989


• CC, Mansour & Godeferd, *J. Fluid Mech.* 337, 1997


• Bellet *et al.*, *J. Fluid Mech.*, 2006

• Scott, *J. Fluid Mech.*, 2013

• Sagaut & Cambon, Homogeneous turbulence dynamics, CUP, New York, 2008