Instability of a plane inertial wave via triadic resonance


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Decaying grid turbulence under rotation

- Turbulence becomes 2D
- Energy transfers between scales are anisotropic

\[
\langle (\delta u)^2 (r) \rangle = \langle (u(x + r) - u(x))^2 \rangle_x
\]

Isotropic turbulence at \( t = 0 \) s

Velocity field

In fluid mechanics under rotation, anisotropic energy transfers between scales (can) proceed via

Triadic resonance of inertial waves
Inertial waves in fluids under rotation

Navier-Stokes equation in a rotating frame

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p - 2\Omega \times u + \nu \Delta u$$

Coriolis force

Restoring action of the Coriolis force

$$\frac{\partial u}{\partial t} = -2\Omega \times u$$

For velocities in the plane \( \perp \Omega \)

$$\Omega = \Omega e_z$$

Anticyclonic circular translation at frequency \( \sigma = 2\Omega \)
Inertial waves in fluids under rotation

Velocity in a plane tilted of $\theta$

$$\mathbf{u} = u_r \mathbf{e}_r + u_y \mathbf{e}_y$$

**Coriolis force**

$$F_c = -2\Omega \times \mathbf{u} = 2\Omega u_y \cos(\theta) \mathbf{e}_r - 2\Omega u_r \cos(\theta) \mathbf{e}_y - 2\Omega u_y \sin(\theta) \mathbf{e}_\theta$$

$$\begin{aligned}
\frac{\partial u_r}{\partial t} &= 2\Omega \cos(\theta) u_y \\
\frac{\partial u_y}{\partial t} &= -2\Omega \cos(\theta) u_r
\end{aligned}$$

Anticyclonic circular translation of the plane tilted of $\theta$

at the frequency $\sigma = 2\Omega \cos(\theta)$
Inertial waves in fluids under rotation

Dispersion relation

→ Propagation along

Cortet, Lamriben, Moisy, Physics of Fluids 22 086603 (2010)
Inertial waves in fluids under rotation

**Coriolis force**

\[ F_c = -2\Omega \times u = 2\Omega u_y \cos(\theta) e_r - 2\Omega u_r \cos(\theta) e_y - 2\Omega u_y \sin(\theta) e_\theta \]

\[ \frac{1}{\rho} \nabla p = 2\Omega u_y \sin(\theta) e_\theta \]

« \( u_0 \cos(\sigma t - \varphi(x)) \) »

\[ \nabla \varphi = \rho \frac{u_0}{\rho_0} 2\Omega \sin(\theta) e_\theta \rightarrow \nabla \varphi \perp u \]

For a monochromatic wave

\[ k = \begin{pmatrix} k \\ m \end{pmatrix} = \nabla \varphi \]

\[ \rightarrow \frac{m}{\kappa} = \frac{\sigma}{2\Omega} \quad \text{and} \quad \kappa = \sqrt{k^2 + m^2} \quad \text{arbitrary} \]

Component along \( e_\theta \) taken by the pressure gradients

Phase velocity \( c_\varphi \parallel \nabla \varphi \)

normal to group velocity \( c_g \parallel \theta \)
Excitation of a plane inertial wave

Plane wavemaker

Gostiaux et al., Exp. in Fluids (2007)
Mercier et al., JFM (2010)
Rotating platform with PIV on board
Excitation of a plane inertial wave


\[ \Omega \]

\[ \theta \]

\[ \Rightarrow \text{Dispersion relation} \]

\[ \Rightarrow \text{Propagation along} \]

\[ \Rightarrow \text{No link between frequency } \sigma \text{ and wavelength} \]

\[ \Rightarrow \text{Phase velocity normal to group velocity} \]
Subharmonic instability

Temporal energy spectrum

\[ E(\sigma) = \langle |\hat{u}_\sigma|^2 \rangle_{x,z} \]

Temporal Fourier transform

\[ \hat{u}_\sigma(x,z) = \frac{1}{2\pi} \int u(x,z,t)e^{i\sigma t} \, dt \]

Primary wave at \( \sigma_0 \)

**Growth of two subharmonic waves**
Temporal Fourier filtering

\[ \frac{\sigma_2}{2\Omega} = 0.59 \]

\[ k = \nabla \varphi \]

The subharmonic waves are plane waves

Primary wave at \( \sigma_0 \)
Triadic resonance of plane waves

Resonance condition for a triad of plane waves

\[ \sigma_1 + \sigma_2 + \sigma_3 = 0 \]

\[ k_1 + k_2 + k_3 = 0 \]

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Triadic resonance of plane waves of inertia

Resonance condition + dispersion relation

\[
\begin{align*}
\sigma_1 + \sigma_2 + \sigma_3 &= 0 \\
k_1 + k_2 + k_3 &= 0
\end{align*}
\]

\[
\frac{\sigma}{2\Omega} = s \frac{m}{\kappa}
\]

avec \( k = \begin{pmatrix} k \\ m \end{pmatrix} \) et \( s = \pm 1 \)

\[
s_0 \frac{m_0}{\sqrt{k_0^2 + m_0^2}} + s_1 \frac{m_1}{\sqrt{k_1^2 + m_1^2}} + \ldots
\]

\[- s_2 \frac{m_0 + m_1}{\sqrt{(k_0 + k_1)^2 + (m_0 + m_1)^2}} = 0.\]

- Experimental triad
Decomposition of Navier-Stokes equation in helical modes

**Helical mode**
\[ u(x, t) = A(t) h_s e^{i(k \cdot x - \sigma_s t)} + \text{c.c.} \]

with
\[ h_s = \frac{k}{|k|} \times \frac{k \times e_z}{|k \times e_z|} + is \frac{k \times e_z}{|k \times e_z|}, \]

→ They constitute a decomposition base for velocity fields

\[ u = \sum_k A_k(k, t) h_s(k) e^{i(k \cdot x - \sigma t)} \]

which allows to recast Navier-Stokes equation in

\[ \left( \frac{\partial}{\partial t} + \nu k^2 \right) A_k = \frac{1}{2} \sum_{p,q} C_{kpq} A_p^* A_q^* e^{i(\sigma_k + \sigma_p + \sigma_q)t} \]

\[ \text{avec} \quad C_k = C_{kpq} = \left[ s_q k_q - s_p k_p \right] (h_{s_p}^* \times h_{s_q}^*) \cdot h_{s_k}^*/2 \]

F. Waleffe, Physics of Fluids A 4, 350 (1992)
A helical mode = a plane inertial wave

Under rotation, if we take $\mathbf{e}_z$ along $\Omega$,

$$u(x, t) = A(t) \ h_s \ e^{i(k \cdot x - \sigma_s t)} + \text{c.c.}$$

avec

$$h_s = \frac{k}{|k|} \times \frac{k \times e_z}{|k \times e_z|} + is \frac{k \times e_z}{|k \times e_z|},$$

describes a plane inertial wave.
Resonant triad of plane inertial waves

Restricting to 3 inertial waves
with initially $A_0 \neq 0$ et $A_{1,2} = 0$

\[
\left( \frac{\partial}{\partial t} + \nu \kappa^2 \right) A_k = C_k A^*_p A^*_q e^{i(\sigma_k + \sigma_p + \sigma_q)t}
\]

(i) The interaction $(k, p, q)$ possible for $k + p + q = 0$

(ii) For $A_{1,2}(t)$ to grow, $\sigma_1 + \sigma_2 + \sigma_3 = 0$

\[
\begin{aligned}
\frac{dA_1}{dt} &= C_1 A_0^* A_2^* - \nu \kappa_1^2 A_1 \\
\frac{dA_2}{dt} &= C_2 A_0^* A_1^* - \nu \kappa_2^2 A_2
\end{aligned}
\]

\[
A_{1,2}(t) = B_{1,2} (e^{\gamma_+ t} - e^{\gamma_- t})
\]

with the growth rate

\[
\gamma_{\pm} = -\frac{\nu}{2} \left( \kappa_1^2 + \kappa_2^2 \right) \pm \sqrt{\frac{\nu^2}{4} \left( \kappa_1^2 - \kappa_2^2 \right)^2 + C_1 C_2 |A_0|^2}
\]
Most instable triad of inertial waves

→ Energy transfers toward modes with more and more horizontal wavevectors \[ \frac{k_{\parallel}}{k_{\perp}} \to 0 \]

→ Mecanism at the origin of the two-dimensionnalisation of rotating turbulence

→ Anisotropic energy transfers between 3 scales
Decaying grid turbulence under rotation

Energy density in the space of scales

\[ E(\vec{r}, t) = \langle (\delta \vec{u})^2 \rangle \]

\[ Re(t) = 1000 \rightarrow 100 \]
\[ Ro(t) = 1 \rightarrow 10^{-2} \]

Turbulence under rotation = Anisotropic energy transfers inside a continuum of scales
Triadic resonance of inertial waves and turbulent energy cascade

\[ \text{Re}_0 = \frac{A_0}{\nu k_0} \gg 1 \quad \Rightarrow \quad \kappa_{1,2} >> \kappa_0 \]

\[ \text{Re}_0 = \frac{A_0}{\nu k_0} \ll 1 \quad \Rightarrow \quad \kappa_{1,2} << \kappa_0 \]

\[ \Rightarrow \text{Direct transfer of energy} \]

\[ \Rightarrow \text{Inverse transfer of energy} \]
Inertial wave turbulence to describe rotating turbulence?

**Capillary wave turbulence**
Herbert, Mordant et Falcon, PRL 2010

\[ E(\sigma, k) \]

\[
\cos \theta = \frac{k_{||}}{|k|}
\]

**Inertial wave turbulence**

\[
\cos \theta = \frac{\sigma}{2\Omega}
\]

\[ \text{WT} \]

\[ \text{« normal » turbulence} \]

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Workshop "non-linear hydrodynamic waves"