The Kelvin-wave cascade in the vortex filament model: Controversy over?

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Classical vs quantum turbulence

Classical theory of turbulence  
Kolmogorov 1941

- Continuous vorticity distribution
- Energy flows to small scales until removed by viscous dissipation
- Constant flux of energy through an inertial range of scales
- Energy spectrum
  \[ E_k = C \varepsilon^{2/3} k^{-5/3} \]

Quantum Turbulence

- Super-cooled Bose gases: e.g. helium\(^4\) below 2.17\(K\)
- Consists of normal and superfluid components
- Bulk superfluid flow is irrotational
- Vorticity confined to 1D topological defects
  Quantized vortex lines
- Vorticity discretized in units of \( \kappa = \hbar/m_4 \)
- QT consists of a tangle of quantized vortices
- K41 energy spectrum still observed at large scales

Polarized vortex bundles mimic classical vortex tubes

Quantum turbulence at small scales

Finite temperature quantum turbulence

- Interlocked normal and superfluid components via mutual friction
- Normal fluid dissipates energy through viscosity
- At zero temperature the normal fluid vanishes

What happens to energy at zero temperature?

- Vortex bundles only make sense at the scales larger than the inter-vortex distance
- Quantized vortices can reconnect exciting Kelvin-waves
- Kelvin-waves propagate along vortex lines with dispersion relation

\[ \omega = -\frac{\kappa k^2}{4\pi} \ln (ka) + C \]

Hypothesis:

Weakly nonlinear Kelvin-wave interactions transfer energy to even smaller scales

Energy dissipation mechanism

- High frequency Kelvin-waves excite phonons that dissipate energy in terms of heat

How do Kelvin-waves interact?
Kelvin-wave turbulence theory

Evolution of quantized vortex lines are governed by the Biot-Savart law

\[ \dot{s} = \frac{\kappa}{4\pi} \int_L \frac{r - s}{|r - s|^3} \times dr \]


- Consider a single, periodic in \( z \), straight vortex line aligned along \( x = y = 0 \)
- Parametrise 2D perturbations by \( s = [x(z), y(z), z] \)
- Define a complex canonical variable: \( w(z) = x(z) + iy(z) \)

\[ i\kappa \frac{\partial w}{\partial t} = \frac{\delta H}{\delta w^*} \quad H = \frac{\kappa^2}{4\pi} \int \frac{1 + \text{Re} \left[ w^*(z_1)w'(z_2) \right]}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}} \, dz_1 \, dz_2 \]

Weak nonlinearity expansion

\[ H = H_2 + H_4 + H_6 + O(\varepsilon^8) \quad \varepsilon = \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|} \ll 1 \]
The six-wave kinetic equation

- There are no nontrivial four-wave resonances
  Leading order are resonant six-wave interactions

\[ \dot{n}_k = \frac{\pi}{6} \int \left| W_{3,4,5}^{k,1,2} \right|^2 n_k n_1 n_2 n_3 n_4 n_5 (n_k^{-1} + n_1^{-1} + n_2^{-1} - n_3^{-1} - n_4^{-1} - n_5^{-1}) \]
\[ \times \delta (k + k_1 + k_2 - k_3 - k_4 - k_5) \delta (\omega_k + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5) \, dk_1 dk_2 dk_3 dk_4 dk_5 \]

Interaction Kernel: \( W_{3,4,5}^{k,1,2} \propto k k_1 k_2 k_3 k_4 k_5 G \)


Non-equilibrium Kolmogorov-Zakharov spectra

1. Constant flux of energy: \( n_k \propto \kappa^{2/5} \epsilon^{1/5} k^{-17/5} \) Kozik-Svistunov spectrum

2. Constant flux of wave action: \( n_k \propto \kappa^{1/5} \eta^{1/5} \Lambda^{-1/5} k^{-3} \)
The four-wave kinetic equation

Locality assumption of wave turbulence theory

- Kolmogorov-Zakharov spectra only exist if wave interactions are local
- Collision integral must be convergent

Collision integral shown to diverge in the limit of two long Kelvin-waves


Local four-wave kinetic equation

\[
\dot{n}_k = \frac{\pi}{12} \int \left\{ |V_k^{1,2,3}|^2 n_1 n_2 n_3 n_k (n_{k_1}^{-1} - n_{k_2}^{-1} - n_{k_3}^{-1}) \delta(k - k_1 - k_2 - k_3) \delta(\omega_k - \omega_1 - \omega_2 - \omega_3) \\
+ 3 |V_1^{k,2,3}|^2 n_1 n_2 n_3 n_k (n_1^{-1} - n_k^{-1} - n_2^{-1} - n_3^{-1}) \delta(k_1 - k - k_2 - k_3) \delta(\omega_1 - \omega_k - \omega_2 - \omega_3) \right\} \, dk_1 dk_2 dk_3
\]

Interaction Kernel: \( V_k^{1,2,3} \propto \Psi k k_1 k_2 k_3 \)

\( \Psi = \frac{1}{\kappa} \int k^2 n_k \, dk \)

New scaling for constant energy flux Kolmogorov-Zakharov solution

\[
n_k \propto \epsilon^{1/3} \Psi^{-2/3} k^{-11/3}
\]

L’vov-Nazarenko spectrum
History of Kelvin-wave simulations


Model

Biot-Savart law (VFM)

Forcing type

Excite vortex line at specific Kelvin-wave frequency

Dissipation type

Smoothing of highest harmonic
History of Kelvin-wave simulations


**Model**
- Biot-Savart Hamiltonian with scale-separation scheme

**Forcing type**
- None (decaying)

**Dissipation type**
- Periodically set high harmonics to zero
History of Kelvin-wave simulations

Boué, Dasgupta, JL, L’vov, Nazarenko, Procaccia, *PRB, 84, 064516, (2011)*

Model

Local-nonlinear equation
(nonlocal limit of Biot-Savart law)

Forcing type
Additive forcing at large-scales

Dissipation type
Large-scale friction and hyper-viscosity

- **Analytical and numerical measurement** of spectrum pre-factor

\[ C_{\text{LN}} = 0.304 \quad n_k = C_{\text{LN}} \epsilon^{1/3} \Psi^{-2/3} \tilde{k}^{-11/3} \]
History of Kelvin-wave simulations


Model
- Gross-Pitaevskii equation

Forcing type
- None (decaying)

Dissipation type
- None but contains phonon emission

\[ k^{-4.116 \pm 0.56} \]

\[ k^{-3.753 \pm 0.17} \]
Why another simulation?

None have been universally accepted in the community

Because...

- Continuing theoretical debate
- Model approximations
- Decaying simulations
- Poor resolution and/or statistics

What is different?

1. Full Biot-Savart simulation without any nonlocal approximations
2. Localised forcing and dissipation (inertial range)
3. True non-equilibrium steady state (forced and dissipated)
4. Long time statistics to distinguish between theoretically predicted spectra
Our vortex filament model setup

\[ \dot{s}_i = \frac{\kappa}{4\pi} \ln \left( \frac{\sqrt{l_i l_{i+1}}}{a} \right) s'_i \times s''_i + \frac{\kappa}{4\pi} \int_{\mathcal{L}'} \frac{r - s_i}{|r - s_i|^3} \times dr + F \]

- **Initial straight vortex line** periodic along \( z \)
- **Third-order Runge-Kutta** time stepping scheme
- **Re-mesh vortex line** onto uniform grid after each time step
- Allows us to **exponentially filter** high and low Fourier harmonics
- **Localized (in Fourier space)** additive forcing

\[ F = [\text{Re}(f), \text{Im}(f), 0], \quad f = \sum_{9 \leq |k| \leq 11} A \exp (ikz + i\phi_k) \]
Results

\[ n_k k^\beta \]

\[ \beta = 11/3 \]
\[ \beta = 17/5 \]
\[ \beta = 3 \]

\[ L \]

\[ |w(z)| \]

\[ -3 -2 -1 0 1 2 3 \]

\[ 0 1 2 3 4 5 \]

\[ 10^{-3} 10^{-2} 10^{-1} 10^0 10^1 10^2 10^3 10^4 10^5 10^6 \]

\[ 0 1000 2000 3000 4000 5000 \]

\[ 6 6.2 6.4 6.6 6.8 7 \]

\[ 0 0.1 0.2 0.3 0.4 0.5 0.6 \]

\[ 0 3 2 1 1 2 3 \]
Conclusions

New simulation of weakly interacting Kelvin-waves in the vortex filament model

• Biot-Savart calculation without approximations
• Non-equilibrium steady state achieved
• Local (in Fourier space) forcing and dissipation (inertial range)

Evidence for nonlocal theory

• Clear distinction between theoretically predicted wave action spectra
• Better agreement with L’vov-Nazarenko spectrum

Spectrum not everything...

• Measurement of energy flux
• Estimate numerical pre-factor of spectrum
• (k, ω)-plot
• Strong wave turbulence
Thank you