JKR theory in detail

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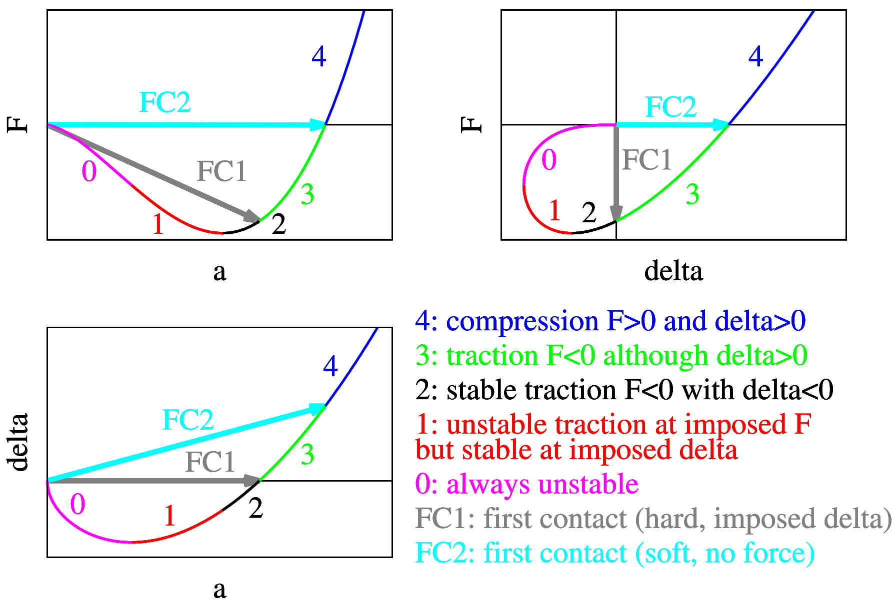

Figure 1: JKR theory: relation between F, a and δ. The specific situations correspond to transitions between different colors: $F = 0$ (blue/green), $\delta = 0$ (green/black), unstability at imposed force or zero rigidity when $F_{\min}$ (black/red), unstability at imposed deflection or zero compliance (red/dashed).

History

The theory of the contact between an elastic sphere and a solid plane (or more generally two elastic spheres of arbitrary radii and elastic moduli) was obtained by Hertz in 1881 [1].

In 1971, Johnson, Kendall and Roberts published a famous article [3] where they present a calculation that includes the effect of the (reversible) energy per unit contact area. Many more calculations and geometries were later presented in a book by Johnson [2].

The calculations for the sphere-plane situation are reproduced below, and a collection of useful formula is given.

Geometry

For two spheres touching each other, their radii R_1 and R_2, Young moduli E_1 and E_2 and Poisson ratios ν_1 and ν_2 must be combined into the two quantities $E^\star$ and R defined by:

$$\frac{1}{E^\star} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \qquad (1)$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (2)$$

Thus, for two identical spheres, one must take $R = R_1/2$ and $E^\star = \frac{1}{2}E_1/(1-\nu_1^2)$, while for a rigid sphere and a deformable plane (or *vice-versa*), one must take $R = R_1$ and $E^\star = E_1/(1-\nu_1^2)$.

In all what follows, a is always the (in-plane) radius of the contact region, while δ is the sum of both deflections in the center of the contact region.

Main calculation

Boussinesq's response to a point force exerted normally onto an elastic half space, and convolution with a pressure field:

$$u = \frac{F}{\pi E^\star r} \qquad (3)$$

$$u = \int\int \frac{p}{\pi E^\star r} \mathrm{d}S \qquad (4)$$

Let us define two pressure fields that will be useful below:

$$p = p_0\sqrt{1-\frac{r^2}{a^2}} \qquad (5)$$

$$p' = \frac{p'_0}{\sqrt{1-\frac{r^2}{a^2}}} \qquad (6)$$

The first pressure field p yields a parabolic displacement inside the contact region of radius a, which is thus suitable to match a sphere surface (to first order):

$$u = \frac{\pi p_0}{4aE^\star}(2a^2 - r^2) \qquad (r < a) \qquad (7)$$

$$u_0 = \frac{\pi p_0 a}{2E^\star} \qquad (r = 0) \qquad (8)$$

$$u = \frac{p_0}{2aE^\star}\left[(2a^2 - r^2)\arcsin\left(\frac{a}{r}\right) + a\sqrt{r^2 - a^2}\right] \qquad (r > a) \qquad (9)$$

The second pressure field p' yields a flat displacement inside the contact region of radius a:

$$u' = u'_0 = \frac{\pi p'_0 a}{E^\star} \qquad (r < a) \tag{10}$$

$$u = \frac{2p'_0 a}{E^\star} \arcsin\left(\frac{a}{r}\right) \qquad (r > a) \tag{11}$$

The elastic energy can be computed as the work of the combined pressure field $p + p'$ over the combined displacement $u + u'$:

$$U_{\mathrm{el}} = \int\int \frac{1}{2}(p + p')(u + u')\mathrm{d}S \tag{12}$$

$$= \frac{\pi^2 a^3}{4E^\star} \int_0^1 x\mathrm{d}x \left[p_0\sqrt{1 - x^2} + \frac{p'_0}{\sqrt{1 - x^2}}\right] \left[p_0(2 - x^2) + 4p'_0\right] \tag{13}$$

$$= \frac{\pi^2 a^3}{4E^\star} \left[p_0(2p_0 + 4p'_0)I(1/2, 1) - {p_0}^2 I(1/2, 3) + p'_0(2p_0 + 4p'_0)I(-1/2, 1) - p_0 p'_0 I(-1/2, 3)\right] \tag{14}$$

where

$$I(a, b) = \int_0^1 (1 - x^2)^a x^b \mathrm{d}x \tag{15}$$

$$I(1/2, 1) = 1/3 \tag{16}$$

$$I(1/2, 3) = 2/15 \tag{17}$$

$$I(-1/2, 1) = 1 \tag{18}$$

$$I(-1/2, 3) = 2/3 \tag{19}$$

Thus:

$$U_{\mathrm{el}} = \frac{\pi^2 a^3}{E^\star} \left[\frac{2}{15}{p_0}^2 + \frac{2}{3}p_0 p'_0 + {p'_0}^2\right] \tag{20}$$

In order to match a sphere of radius R deflecting a plane to depth δ, we must have:

$$u + u' \equiv \delta - \frac{r^2}{2R} \tag{21}$$

From Eqs. (7) and (21):

$$\frac{1}{2R} = \frac{p_0 \pi}{4aE^\star} \tag{22}$$

$$p_0 = \frac{2aE^\star}{\pi R} \tag{23}$$

From Eqs. (8), (10) and (21):

$$\delta = u_0 + u'_0 = \frac{\pi a}{2E^\star}(p_0 + 2p'_0) \tag{24}$$

From Eqs. (23) and (24):

$$p_0' = \frac{E^\star \delta}{\pi a} - \frac{p_0}{2} \tag{25}$$

$$p_0' = \frac{E^\star \delta}{\pi a} - \frac{E^\star a}{\pi R} \tag{26}$$

From Eqs. (20), (23) and (26), we get the elastic energy in terms of a and δ:

$$U_{\rm el} = E^\star a^3 \left[\frac{1}{5}\frac{a^2}{R^2} + \frac{\delta^2}{a^2} - \frac{2}{3}\frac{\delta}{R}\right] \tag{27}$$

To get the total energy, we subtract the interfacial energy:

$$U_{\rm tot} = U_{\rm el} - W\,\pi a^2 \tag{28}$$

To find the radius a of the contact region, we derivate the energy with respect to a at fixed indentation δ:

$$0 = \left.\frac{\partial U_{\rm tot}}{\partial a}\right|_\delta = E^\star \left[\frac{a^4}{R^2} + \delta^2 - 2\frac{a^2\delta}{R}\right] - 2\pi W a \tag{29}$$

In order to reformulate the above equation, we express the applied force as the integral of the total pressure field:

$$F = \int\!\!\int (p + p')\mathrm{d}S \tag{30}$$

$$F = 2\pi a^2 \int_0^1 x\mathrm{d}x \left[p_0\sqrt{1-x^2} + \frac{p_0'}{\sqrt{1-x^2}}\right] \tag{31}$$

$$F = 2\pi a^2 \left[p_0 I(1/2,1) + p_0' I(-1/2,1)\right] \tag{32}$$

$$F = \pi a^2 \left[\frac{2}{3}p_0 + 2p_0'\right] \tag{33}$$

Thus:

$$p_0' = -\frac{1}{3}p_0 + \frac{F}{2\pi a^2} \tag{34}$$

From Eqs. (23), (24) and (34):

$$\delta = \frac{F}{2aE^\star} + \frac{\pi a p_0}{6E^\star} \tag{35}$$

$$\delta = \frac{F}{2aE^\star} + \frac{a^2}{3R} \tag{36}$$

We can now rewrite condition (29) as:

$$\frac{2\pi W a}{E^\star} = \frac{a^4}{R^2} + \delta^2 - 2\frac{a^2\delta}{R} \tag{37}$$

Using Eq. (36):

$$\frac{2\pi W a}{E^\star} = \frac{a^4}{R^2} + \left[\frac{F}{2aE^\star} + \frac{a^2}{3R}\right]^2 - 2\frac{a^2}{R}\left[\frac{F}{2aE^\star} + \frac{a^2}{3R}\right] \tag{38}$$

$$\frac{2\pi W a}{E^\star} = \left[\frac{F}{2aE^\star} - \frac{2a^2}{3R}\right]^2 \tag{39}$$

Main result

Taking the square root of Eq. (39) and choosing the sign such that the interfacial energy W lowers the applied (compressive) force, we obtain an expression for the force in terms of the contact size, then using Eq. (36) we obtain the deflection:

$$F(a) = \frac{4a^3E^\star}{3R} - \sqrt{8\pi W E^\star a^3} \tag{40}$$

$$\delta(a) = \frac{a^2}{R} - \sqrt{\frac{2\pi W a}{E^\star}} \tag{41}$$

Specific situations

When $F = 0$, from Eqs. (40) and (36) we obtain:

$$a_{F=0} = \left(\frac{9\pi}{2}\frac{R^2W}{E^\star}\right)^{1/3} \tag{42}$$

$$\delta_{F=0} = \left(\frac{3\pi^2}{4}\frac{W^2R}{{E^\star}^2}\right)^{1/3} \tag{43}$$

When $W = 0$ (Hertz), from Eqs. (40) and (41) we obtain:

$$F_{W=0}(a) = \frac{4a^3E^\star}{3R} \quad \text{i.e.} \quad a_{W=0}(F) = \left(\frac{3RF}{4E^\star}\right)^{1/3} \tag{44}$$

$$\delta_{W=0}(a) = \frac{a^2}{R} \quad \text{i.e.} \quad a_{W=0}(\delta) = \sqrt{R\delta} \tag{45}$$

$$F_{W=0}(\delta) = \frac{4E^\star R^{1/2}\delta^{3/2}}{3} \quad \text{i.e.} \quad \delta_{W=0}(F) = \left(\frac{9F^2}{16{E^\star}^2R}\right)^{1/3} \tag{46}$$

When $\delta = 0$, from Eqs. (41) then (40) we obtain:

$$a_{\delta=0} = \left(2\pi\frac{R^2W}{E^\star}\right)^{1/3} \tag{47}$$

$$F_{\delta=0} = -\frac{4\pi}{3}WR \tag{48}$$

The limit of stability at zero stiffness (imposed force) is given by the minimum value of the force:

$$F_{\min} = -\frac{3\pi}{2} W R \tag{49}$$

$$a_{F_{\min}} = \left(\frac{9\pi}{8} \frac{R^2 W}{E^\star} \right)^{1/3} \tag{50}$$

$$\delta_{F_{\min}} = -\left(\frac{3\pi^2}{64} \frac{W^2 R}{E^{\star 2}} \right)^{1/3} \tag{51}$$

The limit of stability at infinite stiffness (imposed deflection) is given by the condition:

$$\left. \frac{\partial^2 U_{\text{tot}}}{\partial a^2} \right|_\delta = \sqrt{\frac{32\pi\, W E^\star\, a^3}{R^2}} - 2\pi\, W \geq 0 \tag{52}$$

which happens to be equivalent to the condition:

$$\frac{\mathrm{d}\delta(a)}{\mathrm{d}a} \geq 0 \tag{53}$$

Either of them yields the contact size, then the force and the deflection:

$$a_{\text{instab}-\delta} = \left(\frac{\pi}{8} \frac{R^2 W}{E^\star} \right)^{1/3} \tag{54}$$

$$F_{\text{instab}-\delta} = -\frac{5\pi}{6} W R \tag{55}$$

$$\delta_{\text{instab}-\delta} = -\left(\frac{27\pi^2}{64} \frac{W^2 R}{E^{\star 2}} \right)^{1/3} \tag{56}$$

Note that the tensile force is less tensile than F_{min} and the deflection is more negative than when $F = F_{\text{min}}$.

Alternative expressions

From Eq. (40):

$$a(F) = \left(\frac{9\pi}{8} \frac{R^2 W}{E^\star} \right)^{1/3} \left(1 + \epsilon \sqrt{1 + \frac{2}{3\pi} \frac{F}{W R}} \right)^{2/3} \tag{57}$$

where $\epsilon = 1$ when $a \geq a_{F_{\min}}$ or $\delta \geq \delta_{F_{\min}}$ given by Eqs. (50) and (51) and $\epsilon = -1$ otherwise.

Cubing this equation and expanding the second parenthesis, we find an expression that highlights the difference with the Hertz ($W = 0$) expression:

$$\frac{4E^\star}{3R} a^3(F) = F + 3\pi W R + \epsilon \sqrt{6\pi W R F + (3\pi W R)^2} \tag{58}$$

where $\epsilon = 1$ when $a \geq a_{F_{\min}}$ or $\delta \geq \delta_{F_{\min}}$ given by Eqs. (50) and (51) and $\epsilon = -1$ otherwise.

Then from Eqs. (41) and (57):

$$\delta(F) = \left(\frac{3\pi^2}{4}\frac{W^2 R}{E^{\star 2}}\right)^{1/3}\left(\frac{1+f}{2}\right)^{1/3}\frac{3f-1}{2} \tag{59}$$

$$f = \epsilon\sqrt{1+\frac{2}{3\pi}\frac{F}{WR}} \tag{60}$$

where $\epsilon = 1$ when $a \geq a_{F_{\min}}$ or $\delta \geq \delta_{F_{\min}}$ given by Eqs. (50) and (51) and $\epsilon = -1$ otherwise.

From Eq. (41):

$$\frac{\mathrm{d}\delta}{\mathrm{d}a} = \frac{2a}{R} - \sqrt{\frac{\pi W}{2E^{\star}a}} \tag{61}$$

Repulsive part and adhesive part

From Eq. (33), we define $F = F_0 + F_0'$ with:

$$F_0 = \frac{2}{3}\pi a^2 p_0 \tag{62}$$

$$F_0' = 2\pi a^2 p_0' \tag{63}$$

From Eqs. (23) and (62):

$$F_0(a) = \frac{4a^3 E^{\star}}{3R} \tag{64}$$

Then, from Eqs. (58):

$$F_0(F) = F + 3\pi WR + \epsilon\sqrt{6\pi WRF + (3\pi WR)^2} \tag{65}$$

where $\epsilon = 1$ when $a \geq a_{F_{\min}}$ or $\delta \geq \delta_{F_{\min}}$ given by Eqs. (50) and (51) and $\epsilon = -1$ otherwise.

From Eqs. (26), (41) and (62):

$$F_0'(a) = -\sqrt{8\pi W E^{\star} a^3} \tag{66}$$

Then, from Eqs. (58):

$$F_0'(F) = -\left(3\pi WR + \epsilon\sqrt{6\pi WRF + (3\pi WR)^2}\right) \tag{67}$$

where $\epsilon = 1$ when $a \geq a_{F_{\min}}$ or $\delta \geq \delta_{F_{\min}}$ given by Eqs. (50) and (51) and $\epsilon = -1$ otherwise.

References

[1] H. Hertz. Über die berührung fester elastischer körper. *J. für reine und angewandte Mathematik*, 92:156–171, 1881.

[2] K. L. Johnson. *Contact Mechanics*. Cambridge University Press, Cambridge, New York, Melbourne, 1985.

[3] K. L. Johnson, K. Kendall, and A.D. Roberts. Surface energy and contact of elastic solids. *Proc. Roy. Acad. London A*, 324:301–324, 1971.