Propagation of a transverse wave on a foam microchannel

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Abstract – In a dry foam, soap films meet by three in the liquid microchannels, called Plateau borders, which contain most of the liquid of the foam. We investigated here the transverse vibration of a single Plateau border isolated on a rigid frame. We measured and we computed numerically and analytically the propagation of a transverse pulse along the channel in the 20–2000 Hz frequency range. The dispersion relation shows different scaling regimes, which provide information on the role of inertial and elastic forces acting on the Plateau border. At low frequency, the dispersion relation is dominated by the vibration of the air set into motion by the transverse vibration of the adjacent soap films. The inertia of the liquid in the Plateau border plays a role at high frequency, the critical frequency separating the low-frequency and the high-frequency regimes being a decreasing function of the radius R of the Plateau border.

Introduction . – Liquid foams are dispersions of gas bubbles in a liquid matrix stabilized by surfactants. Due to their diphasic nature, the macroscopic behavior of liquid foams is complex and closely linked to the structure of the liquid skeleton [1]. The acoustic propagation in liquid foams displays such a complex behavior: acoustic resonances and several regimes of propagation have been recently evidenced [2–4], and interpreted as a result of the mechanical coupling between the constitutive elements of the foam skeleton: soap films, liquid channels and the air [4]. However, in order to model the acoustic propagation in a foam, systematic studies of this local coupling must be conducted to identify the local origin of inertia, elasticity and dissipation in a vibrating foam, and the relative roles of the physical characteristics of the bulk liquid, of the gas and of the gas-liquid interfaces.

The liquid network of a dry foam has a well-defined structure. The liquid is contained predominantly in the Plateau border (PB) channels, which form the edges of the faces of the polyhedral bubbles (fig. 1(a)): faces (soap films) meet threefold at 120 degrees in PBs. Each PB is terminated by two vertices in which four PBs meet tetrahedrally [1]. PBs can be isolated on rigid frames, and vibrated by an external forcing in order to study the foam vibration at the scale of the bubbles. Besson et al. [5] and Hutzler et al. [6] have studied oscillating PBs in different geometries. The first study has evidenced the role played by the oscillations around 120 degrees of the contact angle between the soap films, and the second study has suggested that the inertia of the displaced air plays a role in the dynamics of the PB, as described for example in [7] for an oscillating soap film. However, no systematic study of the coupled vibration of a soap film connected to a PB has been performed yet. Several works currently attempt to rationalize the coupled dynamics of the air, the soap films and the PB. Seiwert et al. investigate the case...
of an annular PB bound a soap film and vibrated in a frequency range such as the soap film retains a parabolic shape \[8\]. The free and forced oscillations of the PB and the connected soap film display a coupled dynamics where the displaced air plays a crucial role. In a forthcoming paper, Cohen, Fraysse and Raufaste consider the case of a linear PB, vibrated with a high forcing amplitude: they observe the modulation of the cross-sectional area of the PB in response to the vibration.

In this letter, we consider the propagation of a transverse wave along a linear PB. The PB is long enough so that several wavelengths take place along the PB, and the forcing amplitude is much smaller than the wavelength. We measure the dispersion relation of the wave in the frequency range 20–2000 Hz. By analogy to a vibrating string or membrane, the phase velocity of the wave along the PB can be written as \( c = \omega / q = \sqrt{\gamma / M} \), where \( \omega \) is the angular frequency, \( q \) is the wave number, \( \gamma \) is the air-liquid surface tension and \( M \) is the effective inertial mass per unit area of the vibrating material. We show that \( M \) is a combination of three terms, coming firstly from the mass of the liquid in the vibrating adjacent soap films, secondly from the mass of the air surrounding the films, and thirdly from the mass of the liquid in the PB. In our experimental conditions, the first term is negligible, although it can be taken into account in the model. The thickness of the column of air set into motion by the vibrating soap films is of the order of the wavelength, hence the second term is dominant at large wavelength, that is at low frequency \[7,9\]. Finally, the inertial mass of the liquid in the PB of radius \( R \) becomes significant at high frequency or at large \( R \), where it increases the wave number and lowers the phase velocity of the wave.

**Experimental results.**—Three vertical soap films are suspended on a rigid prismatic frame as shown in fig. 1(c) and meet at the centre of the prism where they form a PB. At equilibrium, the PB is straight and vertical along the \( z \)-axis. We insert into its top vertex \((z = 0)\) the tip of a vertical glass capillary. The capillary is vibrated along the horizontal \( x \)-axis, in the plane of one of the three films meeting at the PB. As a result of capillary forces, the vertex remains attached to the capillary, and a transverse wave propagates along the PB in the \( z \)-direction. The transverse displacement of the PB \( u(z, t) \) along the \( x \)-direction is measured optically and plotted in fig. 2. The wave frame is large and long enough to postpone sufficiently the arrival of the reflected pulse coming from the soap film boundaries or from the other end of the PB: we checked that the signals presented in fig. 2 correspond to the incident propagating pulse alone. Soap solution can be injected at a constant flow rate \( Q \) through the capillary into the PB. The flow rate \( Q \) controls the PB radius \( R \).

Figure 2 shows the propagation of a transverse pulse along the PB. The first extremum of the signal propagates at a velocity of the order of 2 m s\(^{-1}\). However, the deformation of the signal when \( z \) increases shows that the propagation is dispersive. Using a Fourier analysis of the temporal signal \( u(z, t) \), we can determine the complex wave vector as a function of the frequency: \( q = \omega / c - i \Gamma \), with \( \omega = 2\pi f \), and \( c \) and \( \Gamma \) the phase velocity and attenuation of the wave. The imaginary part \( \Gamma \) of the wave vector (not shown here) lies between 10 and 25 m\(^{-1}\) in the whole investigated frequency range. This is at least one order of magnitude lower than the real part of \( q \), therefore \( \Gamma \) is

![Fig. 2: (Color online) Top: space-time diagram of the propagation of a transverse pulse along the PB, for \( f_0 = 75 \) Hz, \( Q = 2 \) ml/min and \( z = 3 \) cm. The Plateau border appears as a black stripe. The width of the stripe along the \( x \)-axis gives a measurement of the PB radius \( R \), as indicated in the figure. The vertical black bar represents 1 mm and the horizontal bar represents 5 ms. Bottom: propagation of the pulse along the PB. Each color corresponds to a different altitude \( z \) below the capillary. The incident pulse is a single undulation starting at \( t = 0 \) and \( z = 0 \).](image)

![Fig. 3: (Color online) Dispersion relation of the transverse wave on the PB for different radii \( R \). The lines correspond to eq. (9) for different \( R \) without any fitting parameter. The typical power laws \( q \sim f^{2/3} \) and \( q \sim f^2 \) are indicated on the graph. Insert: critical frequency \( f_c \) above which \( q(f) \) deviates noticeably from the power law \( q \sim f^{2/3} \) as a function of the PB radius, and comparison to the prediction given by eq. (10).](image)
neglected in the following, and the wave vector \( q \approx \omega/c \) is considered as a real number. The dispersion relation of the transverse wave along the PB is plotted in fig. 3. The parameters used in the experiments are shown in table 1. At low frequency or for small \( R \), all the data collapse on the same master curve, described by a power law \( q \propto f^{2/3} \).

At high frequency and for large \( R \), the data deviate from this power law; the larger the \( R \), the smaller the frequency \( f_s \), separating both regimes. Eventually, we observed that the amplitude of the deformation vanishes in the high-frequency regime (data not shown here), and becomes undetectable at high frequency. Hence, only a small part of this regime can be measured with our experimental technique. This damping is still under investigation and is beyond the scope of this letter.

The vibration of the PB causes the transverse vibration of two of the three adjacent soap films. The dispersion relation of a transverse wave on an infinite soap film is given by [9]:

\[
\frac{\omega^2}{k^2} = \frac{2\gamma}{\rho_m c + 2\rho_a/k}, \tag{1}
\]

where \( k(\omega) \) is the real part of the soap film wave number, \( c \) is the thickness of the soap film and \( \rho_a \) is the mass per unit volume of the surrounding air. The denominator of the right-hand term of eq. (1) corresponds to the mass set into motion by the vibration. In the large wavelength (i.e. small \( k \), or small \( \omega \)) limit, \( 2\rho_a/k \gg \rho_m c \), thus \( k \approx (\rho_a/\gamma)^{1/3}\omega^{2/3} \). This scaling corresponds to the low-frequency regime observed in fig. 3 with \( q = k \): at low frequency, the propagation of the wave on the PB follows the same dynamics as the propagation on the soap film. At higher frequency, the dispersion relation deviates from eq. (1). The transition frequency depends on the PB radius \( R \), suggesting that in this regime, the inertia of the PB starts to play a role. We develop below a model to describe the respective roles played by the liquid channel, the soap films and the surrounding air in the vibration of the PB.

**Model.** – The vibration of the PB occurs along the \( x \)-direction, in the plane of one soap film, and causes an out-of-plane vibration of the other two films called film 1 and film 2. They remain symmetric with respect to the plane \((x, z)\) at all times (see fig. 4(a)). Let \((y_1, z)\) be the plane of film 1, and \(x_1\) the axis normal to this plane, which forms an angle of \( \pi/3 \) with the \( x \)-axis. The transverse deformation of film 1 and film 2 leads locally to a deviation of the angles at the PB from their value at rest \( 2\pi/3 \). Therefore, a net restoring force \( \vec{f} \) due to the surface tension acts on the PB to bring it back to equilibrium (fig. 4(b)) [10]. The propagation of the transverse wave on the PB results from the coupling between the vibration of the PB, of the films and of the surrounding air. Three main simplifications are assumed. First, the role of the longitudinal deformations of the films is neglected and only the transverse deformations are considered. It means that the films are infinitely compressible, with an instantaneous response. This is justified by the low value of the surface elastic modulus of the surfactant solution compared to the surface tension. Second, we assume that the problem is linear, therefore the Fourier modes are not treated different forcing amplitudes. Third, we consider that the soap film has an effective moving mass per unit area \( m_l \) which takes into account the mass of the liquid and the mass of the displaced surrounding air: \( m_l(\omega) = \rho_v c + 2\rho_a/k(\omega) \) where \( k(\omega) \) is the solution of eq. (1). This assumption is valid for a periodic oscillation. The model developed below thus describes the propagation of a single mode of oscillation. Finally, the amplitude of the deformation remains small compared to the wavelength.

Under these assumptions, the equation of the transverse motion of soap film 1 writes

\[
m_l\ddot{\zeta}_1 = 2\gamma (\partial^2_{y_1} + \partial^2_z)\zeta_1. \tag{2}
\]

The inertia of the vibrating PB is balanced by the restoring force \( \vec{f} \) exerted by the soap films pulling on the PB (fig. 4(b)). In the limit of a negligible vertical liquid advection, the equation of the transverse motion of the

<table>
<thead>
<tr>
<th>( Q ) (ml/min)</th>
<th>( R ) (mm)</th>
<th>( e ) (( \mu m ))</th>
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<tbody>
<tr>
<td>0.2</td>
<td>( \leq 0.12 )</td>
<td>0.15</td>
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<tr>
<td>0.5</td>
<td>0.18</td>
<td>0.9</td>
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<tr>
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<tr>
<td>3</td>
<td>0.56</td>
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<tr>
<td>0.7</td>
<td>0.70</td>
<td>2.8</td>
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Table 1: Values of the parameters used in the experiments. The flow rate \( Q \) is a control parameter. The radius \( R \) of the PB is determined using image analysis (see fig. 2); the thickness \( e \) of the soap films is measured using a white light spectrometer (IDIL Fibres optiques - USB2000). The relative errors on \( R \) and on \( e \) are 10%. The differences between several values of \( R \) for the same \( Q \) remain within errors, except for \( Q = 0 \), where the value of \( R \) depends on the time elapsed since the PB has been formed.
PB writes (see appendix B)
\[ \mu \ddot{u} = 2\sqrt{3} \gamma \partial_{y_1} \zeta_1|_{y_1=0}, \]
(3)
where \( \mu = (\sqrt{3} - \pi/2) \rho R^2 = 0.161 \rho R^2 \) is the linear mass of the PB. Since the PB lies at the edge \( y_1 = 0 \) of the soap film, the following boundary condition must be satisfied:
\[ \zeta_1(y_1 = 0, z, t) = \frac{\sqrt{3}}{2} u(z, t). \]
(4)
Equations (2) and (3) are the two equations of motion of the system. Considering a harmonic vibration \( \zeta_1(y_1, z, t) = \zeta_1(y_1, z) e^{i\omega t} \) and a mode of wave vector \( q \) propagating on the PB \( u(t, z) = u_0 e^{i(\omega t - q z)} \), the equations of motion become, taking into account eq. (4):
\[ k^2 \zeta_1 + (\partial^2_{y_1} + \partial^2_z) \zeta_1 = 0, \]
(5)
\[ (\alpha \zeta_1 + \partial_{y_1} \zeta_1)|_{y_1=0} = 0, \]
(6)
where \( \alpha = \mu \omega^2/(3\gamma) \) and \( k \) is the solution of eq. (1). The parameter \( \alpha \) controls the inertia of the PB, whereas \( k \) describes the dynamics of the soap film 1. Solving the whole system analytically requires some assumptions concerning the geometry of the wave (planar or circular) propagating on the soap film. Because the forcing by the capillary is almost point-like, it would generate a circular wave on an infinite soap film. Here, the presence of an inertial PB at the edge of the film deforms the wave front. Therefore, no simple approximation can be inferred concerning the geometry of the wave front. Consequently, we first compute numerically the solutions of eqs. (5) and (6).

**Numerical simulations.** Soap film 1 is approximated by a rectangle and discretized in directions \( y_1 \) and \( z \). The upper boundary at \( z = 0 \) is assumed to be a force-free boundary, i.e. with a vanishing film slope \( \partial_z \zeta_1|_{z=0} = 0 \). The transverse motion of the capillary is imposed by setting the amplitude of the film at \( y_1 = z = 0 \). Equations (5) and (6) are solved numerically, considering a continuous oscillation. To minimize wave reflections at the edges of the film, we included the imaginary part \( k_i \) of \( k \) that reflects the viscous dissipation in the air, as described in appendix A.

The results of the computation are shown in fig. 5 for two values of \( R \), corresponding to two values of the ratio \( \alpha/k \) which compares the inertia of the PB and the inertia of the soap film loaded by the air. Because the forcing is different in the numerical simulations (continuous oscillation) and in the experiments (single burst), the exact value of the transverse amplitude cannot be compared. However, the numerical calculation can be used to visualize the mapping of the deformation in the film. Figure 5 shows that, for \( \alpha \approx 0 \), the wave fronts are circular, as expected for a quasi-point–like perturbation. For \( \alpha \neq 0 \), the circular pattern is deformed close to the PB: a second wave pattern appears, confined along the PB, when the inertia of the PB is finite. In the following, we shall consider that the transverse deformation of the soap film is a linear superposition of a circular wave and of a plane wave localized close to the PB. Under those conditions, an analytical model can be derived.

**Analytical model.** Equations (5) and (6) are solved analytically considering the propagation of a single mode on the PB. We consider two kinds of waves propagating on the soap film: a circular wave and a plane wave.

In the case of a circular wave, the deformation \( \zeta_1 \) is radial. Equation (3) writes, in polar coordinates \((r, \theta) \): \( \mu \ddot{u} = 2\sqrt{3} \gamma r^{-1} \partial_\theta \zeta_1|_{\theta=\pi/2} \), where \( \theta \) is the angle from the \( y_1 \)-axis. Consequently \( \mu \ddot{u} = 0 \): only in the case of an infinitely thin PB or of a vanishing frequency does the wave propagating in the soap film remain circular. If \( \mu \ddot{u} \neq 0 \), the presence of the PB must deform a circular wave in the vicinity of the PB, as illustrated in fig. 5.

Let us now consider the propagation of a plane wave in the soap film: 
\[ \zeta_1(y_1, z, t) = \zeta_0 e^{i(\omega t - q y_1 - q z)} \]
where \( \zeta_0 \) and \( q \) are the components of the wave vector in the soap film. Equation (4) leads to \( q = q_z \) and \( \zeta_0 = (\sqrt{3}/2)u_0 \) and eqs. (5) and (6) lead to:
\[ q_{y_1} = -i\alpha, \]
(7)
\[ q^2 = k^2 + \alpha^2, \]
(8)
Equation (7) means that the deformation \( \zeta_1 \) is damped exponentially in the \( y_1 \)-direction. Equation (8) shows that the wave number along the PB direction is the same as the wave number on the soap film when \( \alpha = 0 \), and increases when \( \alpha \) increases, as depicted in fig. 5. Figure 6 shows that the data extracted from the numerical simulations are very well described by eqs. (7) and (8) in the range of the investigated parameters. This validates the analytical model of a localized plane wave propagating along the PB. Therefore, this analytical model is used in the following to describe the experimental data.

**Comparison with the experiments.** – In the limit of thin liquid films and large wavelength, \( 2\rho u/k \gg \rho c \), the expression of \( k(\omega) \) resulting from of eq. (1) is developed

\[ k(\omega) = \omega \frac{\rho}{\rho + \rho_c} \]
where \( \rho_c \) is the density of the capillary.
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Fig. 6: (Color online) Comparison between the numerical and the analytical dispersion relation, for different $R$. (a) Normalised amplitude $A = \zeta_1(y_1, z_0)/\zeta_1(y_1 = 0, z_0)$ at a fixed $z_0$, indicated as a dashed line in fig. 5. The amplitude decreases exponentially, $A = e^{-\gamma(t)/\xi}$, with a damping factor $\xi$. Insert: $\xi$ vs. $\alpha$ collapse on the first bisector, as predicted by eq. (7). (b) $q/k$ vs. $\alpha/k$, given by the numerical simulations (dots) and by eq. (8) (solid line). The plots corresponding to different $R$ (therefore different $\alpha$) collapse on the same master curve.

The comparison between eq. (9) and the experimental data is shown in fig. 3. The data are very well described by the analytical dispersion relation without any fitting parameter. The low frequency or small $R$ regime corresponds to $q \simeq k_0 = (\rho_u^2/\gamma)^{1/3}$, and eq. (8) becomes

$$q \simeq \left(\frac{\rho_u}{\gamma}\right)^{2/3} \left(\frac{\omega^2}{3}\right)^{1/3} + \left(\frac{\gamma}{3}\right) \omega^2 + \left(\frac{0.161 R^2}{\gamma}\right)^2 \omega^4. \quad (9)$$

The comparison of eq. (9) and the experimental data is shown in fig. 3. The data are very well described by the analytical dispersion relation without any fitting parameter. The low frequency or small $R$ regime corresponds to $q \simeq k_0 = (\rho_u/\gamma)^{1/3}\omega^{2/3}$. In this regime, the dispersion relation of the PB is the same as the dispersion relation of a transverse wave on a soap film, the inertia being dominated by the displaced air. When the frequency or the PB radius increases, the inertia of the liquid in the PB affects the vibration of the soap film edge. Asymptotically, the high frequency or large $R$ regime is dominated by the inertia of the liquid in the PB, according to eq. (9): $q \simeq \alpha \propto R^2 \omega^2$. We did not capture experimentally this asymptotic regime because of the vanishing of the amplitude of vibration of the PB at high frequency. However, we clearly observed a crossover between a low-frequency and a high-frequency regime. According to eq. (9), the crossover between the two regimes happens at a critical frequency $f_c(R)$ such as

$$f_c \simeq \frac{\sqrt{3}}{2\pi} \left(\frac{27\rho_u}{\gamma}\right)^{1/4} \approx 46 \text{ mm}^{-1/2} \text{s}^{-1}. \quad (10)$$

The insert in fig. 3 shows that this prediction reflects reasonably the measured values of the critical $f_c(R)$, with no fitting parameter.

**Discussion and conclusion.** – By mixing experiments, numerical and analytical analysis, we have shown that the propagation of a transverse displacement wave along a linear vertical PB isolated on a frame exhibits two regimes of propagation, in the range 20 Hz–2000 Hz: a low-frequency regime, dominated by the vibration of the adjacent soap films loaded by the air, and a high-frequency regime, where the inertia of the PB dominates.

Several points have to be discussed. Firstly, the vertical liquid velocity has been neglected here. This liquid advection should decrease $q$ and have a larger effect for the larger flow rate. It might explain that, in fig. 3, the data corresponding to $R \geq 0.41$ mm have a lower value than expected and are even almost undistinguishable.

Secondly, the vanishing vibration amplitude, which limits the accessible frequency range at high frequency or large PB is not fully understood. We suspect that it could be due to a low transmission in the intermediate region between the end of the capillary and the PB, where the PB is strongly deformed. However, rationalization of the dissipation in the transient region would be necessary for a complete understanding of this effect, which is beyond the scope of the present letter.

In the future, we plan to measure the attenuation of the signal along the PB using a different technique. Changing the physicochemical composition of the surfactant solution and of the gas should allow us to systematically describe the contributions of the dissipation in the bulk liquid, in the deformable interfaces and in the gas. This should bring new elements to address the more general problem of the dissipation in liquid foams, which is an important and currently active subject of research (see for example the reviews [11,12] and references therein).

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**Appendix A: material and methods.** –

*Experiments.* The PB is created by pulling a rigid plastic wire frame out of a soap solution. The frame (Zometool plastic struts and balls) is made of two horizontal equilateral triangles (width $w = 20 \text{ cm}$) linked at the vertices by vertical beams (height $h = 23 \text{ cm}$). The soap solution is made of distilled water, commercial dishwashing liquid (Fairy Liquid, 1 vol. %) and glycerol (2 vol. %). The mass per unit volume of the solution and of the air are, respectively, $\rho_l = 1003 \text{ kg m}^{-3}$ and $\rho_a = 1.2 \text{ kg m}^{-3}$, the surface tension is $\gamma = 30 \text{ mN m}^{-1}$ and the surface elastic modulus of the soap solution, measured by the pendant drop technique is $|E| = 3 \text{ mN m}^{-1}$ at 1 Hz. The capillary plunging in the PB is connected to a vibrating pot, which generates a controlled vibration along the $x$-axis. The transverse displacement of the PB $u(z, t)$ is recorded using a high-speed camera (Phantom V9) placed at a fixed adjustable height $z$, fitted with a high-magnification objective (Navitar Zoom 6000). The soap solution is injected at a flow rate $Q$ through the capillary using a syringe pump (Pharmacia Biotech P500). $Q$ controls the value of the PB
radius as shown in table 1. We find that for \( Q < 3\) mL/min, the variation of the PB radius \( R(z) \) is smaller than 10% as soon as \( z > 2\) cm (see footnote 1).

Data analysis. We performed Fast Fourier Transform (FFT) \( \hat{u}(z, t) \) of the temporal signal \( u(z, t) \). The FFTs for two different heights, \( z \) and \( z + \Delta z \), are compared. The phase shift \( \phi(t) = \arg[\hat{u}(z + \Delta z, t)] - \arg[\hat{u}(z, t)] \) and the amplitude ratio \( T(\omega) = u(z + \Delta z, t)/u(z, t) \) are extracted. The real and imaginary parts of the complex wave number are then given by \( \omega/c = \phi/\Delta z \) and \( \Gamma = -\ln(T)/\Delta z \). We checked that the measurements are independent of \( \Delta z \), as soon as \( z > 2\) cm. Each signal allowed us to explore a given range of frequencies, around the central frequency of the pulse. Several acquisitions without the capillary, the PB sits at the boundary node \( y_0 \) of every altitude \( y \). The amplitude ratio \( \partial_y \) can be expressed as \( A_{y_0, n} e^{i\omega t} \), where \( y_1 = n_y d_y \) and \( z = n_z d_z \), with \( 0 \leq n_y \leq N_y \) and \( 0 \leq n_z \leq N_z \). The imposed motion of the capillary is taken into account by setting a unit amplitude \( A_{0,0} = A_{0,1} = 1 \). Below the capillary, the PB sits at the boundary node \( n_y = 0 \) at every altitude \( z = n_z d_z \) for \( n_z > 1 \).

The upper boundary at \( n_z = 0 \) is assumed to be a free boundary, i.e. with a vanishing applied force \( \partial_z \xi \) (see appendix B), which can be expressed as \( A_{n_y, -1} - A_{n_y, 0} = 0 \), or more precisely \(-\frac{1}{2} A_{n_y, 0} + 2A_{n_y, -1} + \frac{1}{2}A_{n_y, 2} = 0 \), for all \( n_y > 0 \).

Equation (5) translates into

\[
\begin{align*}
A_{n_y - 1, n_z} + A_{n_y + 1, n_z} + A_{n_y, n_z - 1} + A_{n_y, n_z + 1} - \frac{1}{2} A_{n_y, n_z} &= 0, \\
\left[k^2 - \frac{2}{d_y^2} - \frac{2}{d_z^2}\right] A_{n_y, n_z} &= 0, \quad (A.1)
\end{align*}
\]

and eq. (6) into \( \alpha A|_{y_1 = 0} = \partial_y A|_{y_1 = 0} = 0 \), which becomes

\[
\left( \alpha - \frac{3}{2d_y^2} \right) A_{0, n_z} + \frac{2}{d_y^2} A_{1, n_z} - \frac{1}{2d_y} A_{2, n_z} = 0. \quad (A.2)
\]

At the remote edges \( (n_y = N_y \) and \( n_z = N_z) \), we chose a fixed (vanishing) displacement, i.e. \( A = 0 \). In eq. (A.1), \( k^* = k + ik_z \) is the complex wave vector on the soap film. The real part \( k_z \) is the solution of eq. (1) while the imaginary part \( k_z \) reflects the viscous dissipation in the air as described in ref. [9]: \( k_z = -k\sqrt{\rho_p / (2\omega)} / (\rho_e + 3\rho_p / k) \) where \( \rho_p \) is the air viscosity. With this attenuation, we checked that the exact choice of the boundary conditions at the remote edges does not affect the system essentially in the region of interest. The computation is conducted using the free software GNU octave.

Appendix B: demonstration of eq. (3). – The restoring force \( \bar{f} \) exerted by the soap films pulling on the PB is along the x-axis (fig. 4(b)). Its amplitude is \( f = 2\, \gamma \, (1 - 2\cos\alpha) = 2\, \gamma \, (1 - 2\cos(\pi/3 + \varepsilon)) \) \( \approx 2\sqrt{3}\, \varepsilon \) in the limit of small deformations, where \( 2\alpha \) is the relative angle between film 1 and film 2, and \( \varepsilon \) is the local deviation of film 1 from equilibrium: \( \varepsilon = \partial_{y_1}\xi(y_1 = 0) \) [10]. The inertia of the PB is of the form \( \mu(\partial_t + V_{\xi} \partial_z)\partial_t \), where \( V_{\xi} = Q/S \) is the average vertical liquid velocity in the PB and \( S = (3/2\pi/2)^2 \) is the section of the PB. Considering a single mode of propagation of the wave vector \( q \), the inertia becomes \( -\mu q^2(\omega - q V_{\xi}) \). In the range of the flow rates investigated here, \( V_{\xi} \) varies between 0.09 m/s and 0.17 m/s: it is 10 to 50 times smaller than \( \omega / q \), hence \( V_{\xi} \) can be neglected. Therefore, the equation of the transverse motion of the PB is given by eq. (3).

REFERENCES


1 An expression for \( R(Q, z) \) was derived in ref. [10]. Here, \( R(Q) \) is compatible with a Poiseuille law with mobile interfaces: \( R \approx \left| DnQ/(0.161\, g)\right|^{1/4} \) where \( \nu \) is the kinematic viscosity of the liquid and \( D \) is linked to the interfacial mobility (here \( D \approx 2.9 \)).