Dynamical Solutions of the 3-Wave Kinetic Equations

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Outline

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3-Wave Turbulence

\[ H = T + U = \int \omega_k a_k \bar{a}_k \, dk + \int u(k) \, dk \]

Forcing and dissipation are added to Hamilton’s equations:

\[ \frac{\partial a_k}{\partial t} = i \frac{\delta H}{\delta \bar{a}_k} + f_k - \gamma_k a_k \]

\(a_k, \bar{a}_k\) are complex canonical variables. Interaction energy:

\[ u(k_1) = \int V_{k_1k_2k_3} \left( a_{k_1} a_{k_2} \bar{a}_{k_3} + \bar{a}_{k_1} \bar{a}_{k_2} a_{k_3} \right) \delta(k_1 - k_2 - k_3) \, dk_2 \, dk_3 \]

Scaling parameters:

- Dimension, \(d\): \(k \in \mathbb{R}^d\)
- Dispersion, \(\alpha\): \(\omega_k \sim k^{\alpha}\)
- Nonlinearity, \(\gamma\): \(V_{k_1k_2k_3} \sim k^\gamma\)
The 3-wave kinetic equation

Evolution of WT wave spectrum, $n_k$, given by:

$$\frac{\partial n_{k_1}}{\partial t} = \pi \int V_{k_1 k_2 k_3}^2 (a_1 n_{k_2} n_{k_3} - a_2 n_{k_1} n_{k_2} - a_3 n_{k_1} n_{k_3})$$

$$\delta(\omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \delta(k_1 - k_2 - k_3) \, dk_2 \, dk_3$$

$$+ \pi \int V_{k_2 k_1 k_3}^2 (a_1 n_{k_2} n_{k_3} + a_2 n_{k_1} n_{k_2} - a_3 n_{k_1} n_{k_3})$$

$$\delta(\omega_{k_2} - \omega_{k_3} - \omega_{k_1}) \delta(k_2 - k_3 - k_1) \, dk_2 \, dk_3$$

$$+ \pi \int V_{k_3 k_1 k_2}^2 (a_1 n_{k_2} n_{k_3} - a_2 n_{k_1} n_{k_2} + a_3 n_{k_1} n_{k_3})$$

$$\delta(\omega_{k_3} - \omega_{k_1} - \omega_{k_2}) \delta(k_3 - k_1 - k_2) \, dk_2 \, dk_3$$

$$= a_1 S_1[n_k] + a_2 S_2[n_k] + a_3 S_3[n_k]$$

$a_1 = a_2 = a_3 = 1!$
Isotropic Kinetic Equation: Forward Transfer

Angle averaged spectrum, \( N_\omega = \frac{\Omega_d}{\alpha} \omega^{d-\alpha} n_\omega \), satisfies:

\[
\frac{\partial N_{\omega_1}}{\partial t} = a_1 S_1[N_\omega] + a_2 S_2[N_\omega] + a_3 S_3[N_\omega]
\]

\[
S_1[N_{\omega_1}] = \int L_1(\omega_2, \omega_3) N_{\omega_2} N_{\omega_3} \delta(\omega_1 - \omega_2 - \omega_3) \, d\omega_2 d\omega_3
\]

\[
- \int L_1(\omega_3, \omega_1) N_{\omega_3} N_{\omega_1} \delta(\omega_2 - \omega_3 - \omega_1) \, d\omega_2 d\omega_3
\]

\[
- \int L_1(\omega_1, \omega_2) N_{\omega_1} N_{\omega_2} \delta(\omega_3 - \omega_1 - \omega_2) \, d\omega_2 d\omega_3,
\]

Details are hidden in the kernel \( L(\omega_1, \omega_2) \). Scaling of the interaction coefficient:

\[
L_1(\omega_1, \omega_2) \sim \omega^\lambda, \quad \lambda = \frac{2\gamma - \alpha}{\alpha}
\]
Isotropic Kinetic Equation: Backward Transfer

\[ S_2[N_{\omega_1}] = - \int L_2(\omega_2, \omega_3) N_{\omega_1} N_{\omega_2} \delta(\omega_1 - \omega_2 - \omega_3) \, d\omega_2 d\omega_3 \]
\[ + \int L_2(\omega_3, \omega_1) N_{\omega_2} N_{\omega_3} \delta(\omega_2 - \omega_3 - \omega_1) \, d\omega_2 d\omega_3 \]
\[ + \int L_2(\omega_1, \omega_2) N_{\omega_3} N_{\omega_1} \delta(\omega_3 - \omega_1 - \omega_2) \, d\omega_2 d\omega_3, \]

\[ S_3[N_{\omega_1}] = - \int L_3(\omega_2, \omega_3) N_{\omega_1} N_{\omega_2} \delta(\omega_1 - \omega_2 - \omega_3) \, d\omega_2 d\omega_3 \]
\[ + \int L_3(\omega_3, \omega_1) N_{\omega_2} N_{\omega_3} \delta(\omega_2 - \omega_3 - \omega_1) \, d\omega_2 d\omega_3 \]
\[ + \int L_3(\omega_1, \omega_2) N_{\omega_3} N_{\omega_1} \delta(\omega_3 - \omega_1 - \omega_2) \, d\omega_2 d\omega_3. \]
Physical Meaning of the $S_i[N_\omega]$: Triad Formulation

**Rates:**

- **$S_1[N_\omega]$**
  - Loss $\omega_j$: $\omega_j \cdot L_1(\omega_j, \omega_k) \cdot N_{\omega_j} \cdot N_{\omega_k}$.
  - Loss $\omega_k$: $\omega_k \cdot L_1(\omega_j, \omega_k) \cdot N_{\omega_j} \cdot N_{\omega_k}$.

- **$S_2[N_\omega]$**
  - Gain $\omega_j$: $\omega_j \cdot L_2(\omega_j, \omega_k) \cdot N_{\omega_i} \cdot N_{\omega_j}$.
  - Gain $\omega_k$: $\omega_k \cdot L_2(\omega_j, \omega_k) \cdot N_{\omega_i} \cdot N_{\omega_j}$.

- **$S_3[N_\omega]$**
  - Loss $\omega_j$: $\omega_j \cdot L_3(\omega_j, \omega_k) \cdot N_{\omega_i} \cdot N_{\omega_k}$.
  - Loss $\omega_k$: $\omega_k \cdot L_3(\omega_j, \omega_k) \cdot N_{\omega_i} \cdot N_{\omega_k}$.
Zakharov transformation yields stationary solution:

\[ N_\omega = c_{KZ} \sqrt{J} \omega^{-\frac{\lambda+3}{2}}. \]

where

\[ c_{KZ} = \sqrt{\frac{2}{A}}, \quad A = \left. \frac{dl}{dx} \right|_{x=\frac{\lambda+3}{2}}. \]

and

\[ l(x) = \frac{1}{2} \int_0^1 L_1(y, 1 - y) (y(1 - y))^{-x} (1 - y^x - (1 - y)^x) \]

\[ (1 - y^{2x-\lambda-2} - (1 - y)^{2x-\lambda-2}) \, dy. \]
Numerical Solution of the Isotropic Kinetic Equation

For various reasons one may be interested in more than just the KZ solution. There are no known exact solutions. Discrete case: \( N = (N_1, N_2, N_3, \ldots) \). \( N_i = N(\omega_i) \), \( \omega_i = i\Delta \omega \).

Reduces to a large set of coupled ODEs for \( N \):

\[
\frac{dN}{dt} = S[N] = S_1[N] + S_2[N] + S_3[N]
\]

Numerical solution presents some particular difficulties:

- Widely varying timescales ⇒ use adaptive timestepping.
- System is very stiff ⇒ require *implicit solver*.
- Need to resolve very many modes to measure scaling exponents ⇒ need to approximate the collision integrals.
The 3-Wave Kinetic Equation
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Stiffness: $L(\omega_1, \omega_2) = \omega_1^2 + \omega_2^2, 1000$ modes

Implicit trapezoidal rule (stepwise error of $h^3$):

$$\mathbf{N}(t + h) - \mathbf{N}(t) - \frac{1}{2} h \left[ S[\mathbf{N}(t)] + S[\mathbf{N}(t + h)] \right]$$

Solved via Rosenbrock algorithm.

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3 Wave Kinetic Equation
Computing the Collision Integrals 1

- Divide frequency domain into bins $B_i = [\omega_i^{(L)}, \omega_i^{(R)}]$ having characteristic frequencies $\tilde{\Omega}_i = \frac{1}{2}(\omega_i^{(R)} + \omega_i^{(L)})$ exponentially spaced (except for the first few).
- Apply triad formulation of collision integrals to compute *effective* energy transfer between bins rather than between individual modes.
- $S_1[N]$ requires us to approximate integrals of the form
  \[
  \int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\omega_i \int_{\omega_j^{(L)}}^{\omega_j^{(R)}} d\omega_j (\omega_i + \omega_j) L(\omega_i, \omega_j) N(\omega_i) N(\omega_j)
  \]
- Approximation: if $\tilde{\Omega}_j \leq \tilde{\Omega}_i$ treat all waves in $B_j$ as having frequency $\tilde{\Omega}_j$. (H. Lee 2001)
Thus we obtain one-dimensional integrals which can be done by quadrature:

\[
\begin{align*}
\Delta E_j &= \tilde{\Omega}_j N(\tilde{\Omega}_j) \int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\omega_i, L(\omega_i, \omega_j) N(\omega_i) \\
\Delta E_i &= N(\tilde{\Omega}_j) \int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\omega_i, L(\omega_i, \omega_j) \omega_i N(\omega_i) \\
\Delta E_k &= \Delta E_i + \Delta E_j.
\end{align*}
\]

- Similar expressions for \(S_2[N_\omega]\) and \(S_3[N_\omega]\).
- Many technical details not worth discussing.
Validation of the algorithm

- Stationary scaling exponents are insufficient for validation.
- For some model interactions $c_{KZ}$ can be calculated exactly.
- Product kernel:
  \[ L_1(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\frac{\lambda}{2}}. \]
- Sum kernel:
  \[ L_1(\omega_1, \omega_2) = \frac{1}{2} (\omega_1^\lambda + \omega_2^\lambda). \]

Computing $c_{KZ}$ tests the dynamics.
Example Results

![Graphs showing energy and wave action over time and frequency](image)
It is necessary to truncate the calculation of collision integrals at $\omega = \Omega$: modes having $\omega > \Omega$ have $N_\omega = 0$.

In sum over triads we only include $\omega_j \leq \omega_i < \omega_k \leq \Omega$.

However we must *choose* what to do with triads having $\omega_j \leq \omega_i < \Omega < \omega_k$ (only relevant for $S_1[N_\omega]$).

These terms are included in the sum with weighted by $\nu$:

- $\nu = 1$: open truncation (dissipative)
- $\nu = 0$: closed truncation (conservative)
- $0 < \nu < 1$: partially open truncation (dissipative)

“Boundary conditions” on the energy flux are not local for integral collision operators.
Open Truncation: $\nu = 1$ - Bottleneck Phenomenon

- Product kernel:
  \[ L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}. \]
- Open truncation can produce a bottleneck as the solution approaches the dissipative cut-off (Falkovich 1994).
- Bottleneck does not occur for all $L(\omega_1, \omega_2)$.
- Energy flux at $\Omega$ is 1.

Compensated stationary spectra with open truncation.
Closed Truncation: Thermalisation

- Product kernel:
  \[ L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}. \]

- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).

- Thermalisation occurs for all \( L(\omega_1, \omega_2) \).

- Energy flux at \( \Omega \) is 0.

Compensated quasi-stationary spectra with closed truncation.
Closed Truncation: Thermalisation

- Product kernel:
  \[ L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}. \]
- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).
- Thermalisation occurs for all \( L(\omega_1, \omega_2) \).
- Energy flux at \( \Omega \) is 0.

Bare quasi-stationary energy spectra with closed truncation.
Finite and Infinite Capacity Cascades

Stationary KZ spectrum:

\[ N_\omega = c_{KZ} \sqrt{J} \omega^{-\frac{\lambda+3}{2}}. \]

Total energy contained in the spectrum:

\[ E = c_{KZ} \sqrt{J} \int_1^\Omega d\omega \omega^{-\frac{\lambda+1}{2}}. \]

- \( E \) diverges as \( \Omega \to \infty \) if \( \lambda \leq 1 \) : Infinite Capacity.
- \( E \) finite as \( \Omega \to \infty \) if \( \lambda > 1 \) : Finite Capacity.

Transition occurs at \( \lambda = 1 \).
Finite capacity systems exhibit a dissipative anomaly in the usual sense:

\[ \lambda = \frac{3}{4} \]

\[ \lambda = \frac{3}{2} \]
Dynamical Scaling

Self-similarity ansatz describing the establishment of the K–Z spectrum (Falkovich and Shafarenko, 1991):

\[ N(\omega, \tau) = \tau^a F(\eta) \]

propagating front with power law “wake”.

\[ \eta = \frac{\omega}{\tau^b} \quad \tau = t \quad \text{Infinite capacity case} \]

\[ \tau = t^* - t \quad \text{Finite capacity case} \]

Dynamical scaling exponents, \(a\) and \(b\).

\[ x = -\frac{a}{b} \]

is the exponent of the wake.
Dynamical Scaling

Self-similarity requires: \( a + (\lambda + 1)b = -1 \).

Profile \( F \) determined by integro-differential equation:

\[
\pm b \eta \frac{dF}{d\eta} - aF = S_1[F(\eta)] + S_2[F(\eta)] + S_3[F(\eta)]
\]

Infinite capacity case:

- Total energy \( E \sim t \Rightarrow 2b + a = 1 \).
- \( a = \frac{\lambda + 3}{\lambda - 1} \), \( b = -\frac{2}{\lambda - 1} \)
- \( x = \frac{\lambda + 3}{2} \) which is the K–Z exponent.

Finite capacity case:
- ?
Measuring dynamical scaling exponents

- Finite capacity singularity \((t - t^*)^b\) makes direct fitting difficult.
- An easier measurement:

\[
\begin{align*}
M_n(t) &= \int \omega^n N_\omega \\
R_n(t) &= \frac{M_{n+1}(t)}{M_n(t)} \\
G_n(t) &= \frac{R_n(t)}{\dot{R}_n(t)}
\end{align*}
\]

Measuring dynamical scaling via \(G_3\).

Self-similarity ansatz \(\Rightarrow G_n(t) \sim \frac{1}{b}(t - t^*)\). (fit a linear function).
Is there a dynamical scaling anomaly?

- Transient spectrum is often steeper than $x_{KZ}$ for finite capacity cascades? (Galtier et al. (2000), Lee (2000), CC, Newell and Pomeau (2003), CC and Nazarenko (2004))

- This phenomenon is not well understood.

- If there is an anomaly in general, it is very small.
Cascades without back-scatter

The kinetic equation without backscatter:

$$\frac{dN}{dt} = S_1[N].$$

What are the effects of removing backscatter?

- No thermalisation.
- Bottleneck phenomenon remains.
- A new type of singular solution emerges.
“Anomalous Dissipative Anomaly” (Dynamic Nonlocality)

Decay problem: \[ L_1(\omega_1, \omega_2) = \omega_1^\lambda + \omega_2^\lambda \]

For \( \lambda > 1 \), as \( \Omega \to \infty \), \( t^* \to 0 \) removing all energy.
“Reset” Phenomenon and Nonlocal Oscillations

- Dynamic nonlocality in the presence of a source leads to oscillatory behaviour.
- In such situations, there is no stationary state, no self-similarity.
- Unclear whether this phenomenon exists in the full 3WKE (can backscatter beat nonlocality?)

\[ L(\omega_1, \omega_2) = \omega_1^{1+\varepsilon} + \omega_2^{1+\varepsilon} \]
Conclusions

- New numerical method for solving the isotropic 3-wave kinetic equation
- Choice of spectral truncation allows one to produce bottleneck and/or thermalisation phenomena.
- Finite capacity systems exhibit a dissipative anomaly in the usual sense.
- Dynamical scaling exponents can be measured and do not show a strong dynamical scaling anomaly (at least for the product kernel).
- Removal of backscatter terms from the kinetic equation produces surprising new phenomena which suggest the scaling theory of the full kinetic equation may also contain hidden surprises.