Wave Attractors in stratified & rotating media

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Summary

• Geo & astrophysical fluids (ocean, atmosphere, planets, stars): external & internal waves
• In confined, *symmetry-breaking* fluid domains, *linear* internal waves are (generally) *multiscale*

*wave attractor*  
Kopecz 2004
Linear waves

\[ \omega(k) \]

\[ \omega(\kappa) \]  discrete

\[ \omega(\theta) \]  continuous

**Geo- & Astrophys.**  Spatial structure  Dispersion Rel.  Spectrum

External (surface)  elliptic PDE

Internal  hyperbolic PDE

Frequency  \( \omega \)

Wavenumber vector  \( k = \kappa(\cos \theta, \sin \theta) \)
Surface gravity waves

Dispersion relation \( \omega^2 = g\kappa \tanh(\kappa h) \), \( \kappa = |\mathbf{k}| \)

Chaotic wave rays
Helmholtz Eq. – Geom. Optics
Quantum chaos

‘Scarring’
Pointillistic → monolongitudinal
single scale
# Internal waves

**Fluid stratified in**
- density
- angular momentum
- ?

**Restoring force**
- Gravity (buoyancy)
- Coriolis
- Lorentz

**Dispersion Rel.**
- \[ \omega = \frac{N \cos \theta}{2} \]
- \[ \omega_c = \frac{eB}{m} \]

## Maximum frequency:

<table>
<thead>
<tr>
<th>Type</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buoyancy</td>
<td>[ N = \left( \frac{-g d\rho_0}{\rho_* dz} \right)^{1/2} ]</td>
</tr>
<tr>
<td>Inertial (Coriolis)</td>
<td>2( \Omega )</td>
</tr>
<tr>
<td>Electron-cyclotron</td>
<td>[ \omega_c = \frac{eB}{m} ]</td>
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</tbody>
</table>
Internal gravity waves

Uniform stratification $N = \text{constant}$

$\frac{1}{2} < \frac{\omega}{N} < 1$

$\frac{\omega}{N} < \frac{1}{2}$

$\omega = \omega(\theta)$

$c_g = \nabla_k \omega \perp c = \frac{\omega}{|k|^2}k$

Sakai, Iizawa, Aramaki 1997
Internal wave spatial structure

Streamfunction $\Psi$ obeys spatial wave equation in stretched coordinates: $$\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} = 0$$

Free waves: $\Psi = 0$ at all boundaries $\rightarrow$ simplest nontrivial 2\textsuperscript{nd} order BVP

Forced problems: $\Psi = \Psi(s)$, $s$: along-boundary coordinate

$\Psi = f(X + Z) - g(X - Z)$

Characteristics $x \pm z = \text{const}$
Arbitrarily shaped domains

Streamfunction $\Psi = f(X+Z) - g(X-Z)$

pressure $p = f(X+Z) + g(X-Z)$

$f^* is invariant of web of connected characteristics. f & g are partial pressures.$

$X+Z = c_1, \quad X - Z = c_2$

$\Psi = 0 \rightarrow g^* = f^*$
Exact geometric solution:

\[ \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} = 0 \]

Fundamental intervals: give \( \Psi_z \)

Wave attractor

Characteristics: \( x \pm z = \text{const.} \)

\[ z = \frac{3}{2} (x - 1) \]

Self-similar but, .. no analytic derivatives

BC: \( \Psi = 0 \)

Specific frequency: \( z = -3/2 \)

\( x = -1 \)

\( x = 0 \)

\( x = 1 \)

John 1941
Maas & Lam 1995
Analytic Fourier collocation

Solve: \[ \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} = 0 \]

Multiscale free-wave solution \[ \Psi = \sum_{n=1}^{\infty} a_n \sin 2n\pi \frac{(x+1)}{3} \sin 2n\pi \frac{z}{3} \]

\[ a_{2n+1} = 0 \]

Functional equation
Exact analytic Fourier amplitudes

$$E_m = \langle u^2 + w^2 \rangle = m^2 a_{2m}^2$$

Logarithmic self-similar multiscale spectrum

$$a_{2m} = \frac{2m(-1)^m}{\pi} \sum_{n=0}^{\infty} \sin \left( \frac{m\pi}{2b_n} \right) \left( \frac{1}{m^2 - b_n^2} - \frac{1}{m^2 - b_{n+1}^2} \right) , \quad b_n \equiv \frac{3}{2} \cdot 5^n$$

Related to Weierstrass function

Maas 2009
Generation wave attractor - laboratory observation

Buoyancy: \[ b = -\frac{g\rho'}{\rho_*} \]
\[ b \propto w = \psi_x \]

Color:

Perturbation vertical buoyancy gradient \[ A = b_z / N \]
1 frame/period

oscillation

*Hazewinkel et al 2008*
Decay phase, 8 frames/period
Spectral development

Observed multiscale Internal Wave field stationary phase

Spectral development
Rotating fluids: Poincaré Eqn.

Henri Poincaré, *Acta Mat.* t.7 (1885)
Sur l’équilibre d’une masse fluide animée d’un mouvement de rotation

Marcel Brillouin, *Annales de l'I.H.P.*, tome 1, 3 (1930)
Quelques propriétés d'une équation aux dérivées partielles hyperbolique
Inertial wave equations

\[ \begin{align*}
    u_t - fv &= -p_x \\
    v_t + fu &= -p_y \\
    w_t &= -p_z \\
    u_x + v_y + w_z &= 0
\end{align*} \]

Monochromatic wave: frequency $\omega < f = 2\Omega$

Spatial structure: (hyperbolic) Poincaré Eqn.:

\[ p_{xx} + p_{yy} - \left( \frac{f^2}{\omega^2} - 1 \right) p_{zz} = 0 \]

Plane wave dispersion relation:

\[ \left( \frac{\omega}{f} \right)^2 = \frac{m^2}{k^2 + l^2 + m^2} = \sin^2 \theta \]

At boundary: $u.n = 0$

oblique-derivative BCs: $a p_\parallel + b p_\perp = 0 \quad \rightarrow \text{multiscale}$

Flat box: vertical dependence separable
Solve generalized eigenvalue problem

Energy (kinetic + potential) of some modes in *flat* rotating cube

No attractors, yet multiscale and degenerate

Rectangular geometrical vs cylindrical rotational symmetries

*Maas 2003*
Inertial wave experiments

Breaking axial symmetry
Observed harmonic amplitudes

Wave attractors for different $\omega$ in 5 m long basin

Manders & Maas 2003
Attractor mixes angular momentum
dye PIV \((u_0, v_0)\)
slope
Topview generates cyclonic mean flow

Maas 2001
Geo & Astrophysical relevance

Wave attractors in homogeneous, rotating spherical shell
Stern 1960, Bretherton 1964, Stewartson 1971
- Earth’ liquid outer core Rieutord et al 2001, Tilgner 1999
- Stellar convective interior Ogilvie & Lin 2004, Ogilvie 2009, Rieutord and Valdettaro 2009

Ogilvie 2009
Harlander & Maas 2007
Nonlinearity?

- Triangle of interaction yields higher harmonics
- Wave breaking and mixing yield mean field effects
- Linear wave’s spatial structure determined by ‘web’ of characteristics

\[(u \cdot \nabla)u = (u \cdot k)u = 0\]
Parabolic channel: $H(x) = \tau(1-x^2)$

Geometry of characteristics

Irrespective of particular $x_0$, characteristics approach limit cycle: wave attractor

$Z = -\tau$

$t = 0.9$

**Bird’s eye view**

Surface reflections of attractor for varying ‘depths’ $\tau = \left( \frac{N}{\omega^2} - 1 \right)^{1/2} \frac{D}{L}$

**No** regular eigenmodes; singularity on wave attractor

*Maas & Lam 1995*
Chaos?

• No, Lyapunov exponent $\lambda_+ \leq 0$
• Remarkable selfsimilarity

-Dissipative, orientation preserving nonlinear map of circumference onto itself

-Devil’s staircase in rotation number

-Arnol’d tongues

*Manders et al 2003*
Conclusion

• Linear internal waves in geo & astrophysics are multiscale – related to wave attractors
• Wave attractors: self-similar in physical and parameter space
• Geometric structure of linear wave pattern determined by nonlinear map of circumference onto itself
• Dynamic nonlinearities localized to attractor
• Internal wave attractor: challenge to wave turbulence