Chaos in diffusive systems

Tanguy Laffargue[1,2] (tanguy.laffargue@univ-paris-diderot.fr), Peter Sollich[1] (peter.sollich@kcl.ac.uk), Frédéric van Wijland[2] (fvw@univ-paris-diderot.fr)

1. King's College London   2. Laboratoire Matière et Systèmes Complexes, Université Paris Diderot-Paris 7

Abstract

We are looking for the presence of dynamical phase transitions in diffusive systems which are described by fluctuating hydrodynamics.

- We look at the Lyapunov spectrum to identify a change of dynamical regime.
- We generalise the approach used on discrete systems to isolate the largest Lyapunov exponent.
- We use the Martin-Siggia-Rose-Jansen-de Dominicis (MSRJD) formalism to express this problem in terms of a path integral.
- We apply the Saddle-Point approximation to evaluate the large deviation function for the largest exponent.
- We look at damage spreading in the discrete version of those systems and find a mapping to reaction-diffusion processes.

Introduction

Systems with diffusive dynamics are many-body interacting particle systems that can be studied by means of a noisy diffusive equation governing the dynamics of a single conserved density. Among such systems, the celebrated Symmetric Simple Exclusion Process (SSEP) is famous for serving as a test bench for many ideas in non-equilibrium statistical mechanics. Our focus is interplay of the Lyapunov spectrum properties with the existence of dynamical phase transitions.

Lyapunov spectrum

The Lyapunov spectrum is a measure of the sensitivity to initial conditions. The Lyapunov exponents characterise the exponential separation of infinitesimally close trajectories. Consider a system described by a vector x of dimension D, which evolves according to the equation:

$$\dot{x}(t) = f(x(t))$$

where f is the evolution operator of the system. Let u be a small perturbation of a solution x to this equation, such that $x+u$ is still a solution of this equation. To first order in u, we have:

$$\dot{u}(t) = Jf(x(t))\ u(t)$$

where $Jf(x)$ is the Jacobian matrix of f at x. If we denote by $A(t)$ the matrix solution of

$$\dot{A} = Jf(x(t))A$$

we can solve the previous equation:

$$u(t) = A(t)\ u(0)$$

We are interested in the divergence of the norm of the perturbation. We have:

$$|u(t)| = \left[u^\dagger(0)A^\dagger(t)A(t)u(0)\right]^{\frac{1}{2}}$$

The Lyapunov exponents at finite time t are defined as the eigenvalues of the operator $\frac{1}{2t}\ln\left(A^\dagger(t)A(t)\right)$. When the spectrum $\{\lambda_i(t)\}$ has a finite limit for $t \to +\infty$, we can define the Lyapunov exponents $\{\lambda_i\}$ by $\lambda_i = \lim_{t\to\infty} \lambda_i(t)$.

If the Lyapunov spectrum exists, we can isolate the largest Lyapunov exponent by:

$$\lambda_1 = \lim_{t\to+\infty} \frac{1}{t}\ln\frac{|u(t)|}{|u(0)|}$$

We can now define the largest Lyapunov exponent at finite time t by:

$$\lambda_1(t) = \frac{1}{t}\ln\frac{|u(t)|}{|u(0)|}$$

This definition is not equivalent to the previous one, but it gives the same results in the $t \to +\infty$ limit and it is easier to calculate.

Fluctuating hydrodynamics

Fluctuating hydrodynamics is a description of a diffusive systems at a macroscopic scale. This formulation is more intuitive; instead of looking at each particle individually, we are looking at the density ρ of particles in the system. The time evolution of ρ is a continuity equation. The current which appears in this equation is fluctuating. In the general case, a one dimensional system of size L is described, when the size L goes to infinity, by the equation:

$$\partial_\tau \rho(x,\tau) = -\partial_x j(x,\tau)$$

with:

$$j(x,\tau) = -D(\rho(x,\tau))\ \partial_x\rho(x,\tau) - \sqrt{\frac{\sigma(\rho(x,\tau))}{L}}\ \xi(x,\tau)$$

Mapping between damage spreading and diffusive systems

When we want to study damage spreading in a stochastic system, there is an important question: what does it mean for two copies of a system to evolve according to the same equation. This question is trivial for a system described by a Langevin equation: the two copies need to have the same noise. But for a discrete process, like the SSEP or free particles on a network, it is quite hard to define applying the "same noise". For a discrete system of particles, we want the two systems to remain identical if they are identical initially. More precisely, we want the identical parts of the two systems to stay identical. So we only care about the particles which have no equivalent in the other system. With this is mind, we can show that damage spreading for free particles on a chain is equivalent to the reaction-diffusion process $A+B\to\emptyset$ with an infinite reaction rate, while for the SSEP it is equivalent to the same process, but with exclusion between the A and B particles. The A particles represent the particles in the first system with no equivalent in the second system and the B particles the opposite. The total density for this process decays with a power law, that is to say with a Lyapunov exponent equal to zero. How can we link this to fluctuating hydrodynamics?

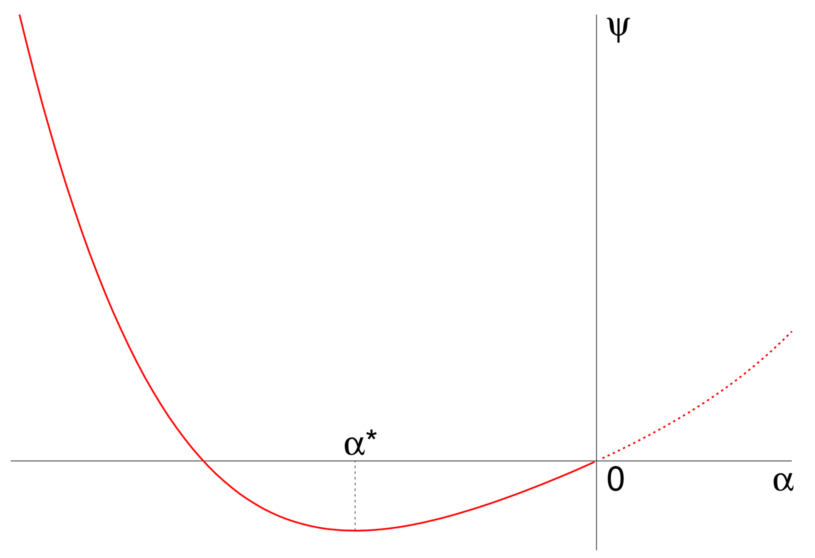
Saddle-Point calculation: results

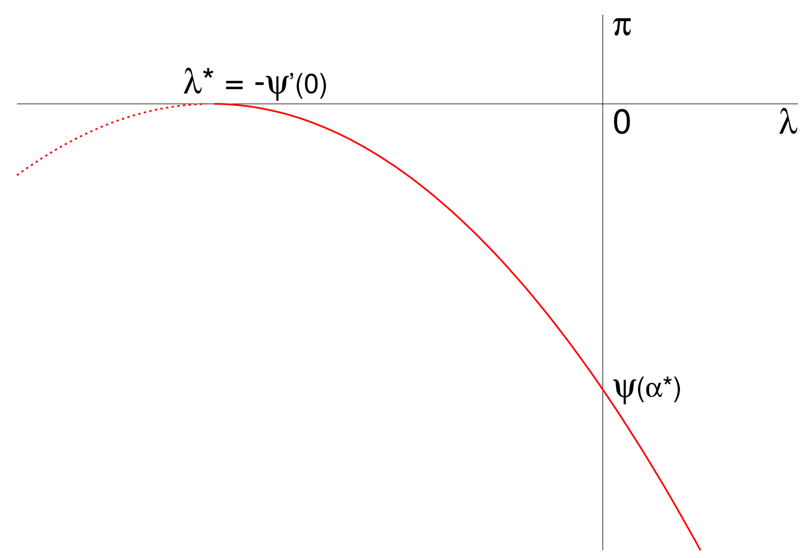
By using the Saddle-Point approximation and a perturbative expansion in α, we can show for the three models considered:

$$\psi(\alpha) = 4\pi^2\alpha\ \mathcal{F}\left(\frac{\sigma'(\rho_0)^2}{\sigma(\rho_0)}\alpha\right) \text{ for } \alpha < 0$$

with:

$$\mathcal{F}(x) = 1+\frac{1}{8}x+\frac{3}{128}x^2+\frac{55}{4096}x^3+\mathcal{O}(x^4)$$





The first term of j is deterministic. It corresponds to Fick's law. It controls the dynamics of system on a macroscopic scale. The second term is stochastic. At the macroscopic scale, the fluctuations are insignificant. The function σ is a phenomenological coefficient, which should vanish when the density is zero.

In this equation, $\tau = \frac{t}{L^2}$ is the diffusive time, $x \in [0,1]$ is the position variable and ξ is a white Gaussian noise of unit variance. The functions $D(\rho)$ and $\sigma(\rho)$ depend on the model. We are going to consider three models: the free particles, the Symmetric Simple Exclusion Process (SSEP) and the Kipnis-Marchioro-Presutti model (KMP). For those models, D and σ are:

- Free particles: $D(\rho) = 1$ and $\sigma(\rho) = 2\rho$
- SSEP: $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$
- KMP: $D(\rho) = 1$ and $\sigma(\rho) = 4\rho^2$

When we are dealing with a Langevin equation with multiplicative noise, we need to specify if it is an Itô equation or a Stratonovitch one. Here it is an Itô equation. Since we are considering the large L limit, the nature of the stochastic equation is not relevant here.

Largest Lyapunov exponent

We consider a one dimensional system of size L with periodic boundary conditions which evolves according to fluctuating hydrodynamics with $D = 1$.

$$\partial_\tau\rho(x,\tau) = \partial_x^2\rho(x,\tau) + \partial_x\left(\sqrt{\frac{\sigma(\rho(x,\tau))}{L}}\xi(x,\tau)\right)$$

Let ρ be the solution of this equation with mean density ρ_0, and let u be an infinitesimal perturbation of the solution, such that $\rho+u$ is still a solution of mean density ρ_0.

We define $|u|(\tau) = \left[\int_0^1 \mathrm{d}x\ u^2(x,\tau)\right]^{1/2}$ and $v(\tau) = \frac{u(\tau)}{|u|(\tau)}$. We can show:

$$|u|(\tau) = e^{\tau\lambda(\tau)}\,|u|(0)$$

where $\lambda(\tau)$ is the largest Lyapunov exponent at finite time τ, given by:

$$\lambda(\tau) = \frac{1}{\tau}\int_0^\tau \mathrm{d}t \int_0^1 \mathrm{d}x \left[v(x,t)\ \partial_x^2 v(x,t) + v(x,t)\ \partial_x\left(\frac{\sigma'(x,t)}{2\sqrt{L\ \sigma(\rho(x,t))}}\ v(x,t)\ \xi(x,t)\right)\right]$$

Large deviation and MSRJD formalism

We are interested in the probability distribution $P(\lambda,\tau)$ of the largest Lyapunov exponent λ at time τ. The large deviation formalism allows us to express it as:

$$P(\lambda,\tau) \underset{\substack{\tau\to\infty \\ L\to\infty}}{\sim} e^{L\tau\pi(\lambda)}$$

Instead of directly working with the probability distribution, we can consider the moment-generating function . We define:

$$Z(\alpha,\tau) = \left\langle e^{-\alpha L\tau\lambda(\tau)}\right\rangle \underset{\substack{\tau\to\infty \\ L\to\infty}}{\sim} e^{L\tau\psi(\alpha)}$$

where α is the conjugate parameter to λ. Depending on its value, it biases the measure by favoring abnormally stable trajectories ($\alpha > 0$) or highly chaotic trajectories ($\alpha < 0$). ψ is the Legendre transform of π :

$$\psi(\alpha) = \max_\lambda\left(\pi(\lambda) - \alpha\lambda\right)$$

Using the MSRJD formalism, we can express $Z(\alpha,\tau)$ as a path integral.

Outlook

In future work we plan to:

- Try to understand what happens for $\alpha > 0$.
- Look at damage spreading in the discrete version of those models.
- Study more precisely the link between damage spreading in the discrete models and the reaction-diffusion processes.